FLAT CONNECTIONS ON CONFIGURATION SPACES AND FORMALITY OF BRAID GROUPS OF SURFACES

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Abstract. We construct an explicit bundle with flat connection on the configuration space of n points of a complex curve. This enables one to recover the 'formality' isomorphism between the Lie algebra of the prounipotent completion of the pure braid group of n points on a surface and an explicitly presented Lie algebra $\mathfrak{t}_{g,n}$ (Bezrukavnikov), and to extend it to a morphism from the full braid group of the surface to $\exp(\hat{\mathfrak{t}}_{g,n}) \rtimes S_n$.

Introduction

One of the achievements of rational homotopy theory has been a collection of results on fundamental groups of (quasi-)Kähler manifolds, leading in particular to insight on the Lie algebras of their prounipotent completions ([Su, Mo, DGMS]; for a survey see [ABCKT]). These results are particularly explicit in the case of configuration spaces $X = \mathrm{Cf}_n(M)$ of n distinct points on a manifold M ([Kr, FM, To]). In the particular case where M is a compact complex curve, they were made still more explicit in [Bez] (see also [Ko] for the case $M = \mathbb{C}$). In these works, a 'formality' isomorphism was established between this Lie algebra, denoted $\mathrm{Lie}\,\pi_1(X)$, and an explicit Lie algebra $\hat{\mathfrak{t}}_{g,n}$, where g is the genus of M ($\hat{\mathfrak{t}}_n$ when $M = \mathbb{C}$).

All these works take place in the framework of minimal model theory. However, alternative proofs are sometimes possible, based on explicit flat connections on X. Through the study of monodromy representations, such proofs allow for a deeper study of the algebra governing the formality isomorphisms, as well as for their connection to analysis and number theory.

In the case $X = \mathrm{Cf}_n(\mathbb{C})$, a construction of the formality isomorphism $\mathrm{Lie}\,\pi_1(X) \simeq \hat{\mathfrak{t}}_n$, based on a particular bundle with flat connection on X, can be extracted from [Dr]. This flat connection is at the basis of the theory of associators developed there; when certain Lie algebraic data are given, it specializes to the Knizhnik-Zamolodchikov connection ([KZ]). When $X = \mathrm{Cf}_n(C)$, where C is an elliptic curve, a bundle with flat connection over X was constructed in [CEE] (see also [LR]) and an isomorphism $\mathrm{Lie}\,\pi_1(X) \simeq \hat{\mathfrak{t}}_{1,n}$ was similarly derived; this flat connection specializes to the elliptic KZ-Bernard connection ([Ber1]). The corresponding analogue of the theory of associators was later developed by the author.

The goal of the present paper is to construct a similar explicit bundle with flat connection over $X = \mathrm{Cf}_n(C)$, C being a curve of genus ≥ 1, and to derive from there an alternative construction of the isomorphism of [Bez]. We first recall this isomorphism (Section 1). We then recall some basic notions about bundles and flat connections in Section 2, and we formulate our main result: the construction of a bundle $\mathcal{P}_n$ over X with a flat connection α_{KZ} (Theorem 3), in Section 3. There we also show (Theorem 4) how this result enables one to recover the isomorphism result from [Bez], as well as to extend it to a morphism from the full braid group in genus g to $\exp(\hat{\mathfrak{t}}_{g,n}) \rtimes S_n$. Section 4 contains the explicit construction of the connection α_{KZ}. The rest of the paper is devoted to the proof of its flatness. Section 5 is a preparation to this proof, and studies the behaviour of α_{KZ} under certain simplicial homomorphisms. Section 6 contains the main part of the proof, while Section 7 contains the proof of some algebraic results on the Lie algebras $\mathfrak{t}_{g,n}$ which are used in the previous section.

We hope to devote future work to applications of the present work to a theory of associators in genus g, as well as to relation with the higher genus KZB connection ([Ber2]).

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1. Formality results

Let $g \geq 0$ and $n > 0$ be integers. The pure braid group with n strands in genus g is defined as $P_{g,n} := \pi_1(\mathrm{Cf}_n(S), x)$, where S is a compact topological surface of genus g without boundary, $\mathrm{Cf}_n(S) = S^n -$ (diagonals) is the space of configurations of n points in S, and $x \in \mathrm{Cf}_n(S)$. The corresponding braid group is $B_{g,n} = \pi_1(\mathrm{Cf}_{[n]}(S), \{x\})$, where $\mathrm{Cf}_{[n]}(S) = \mathrm{Cf}_n(S)/S_n$ and $\{x\}$ is the S_n-orbit of x.

If $g > 0$ and $n \geq 0$, define $\mathfrak{t}_{g,n}$ as the $\mathbb{C}$-Lie algebra with generators[1] v^i ($v \in V$, $i \in [n]$), t_{ij} ($i \neq j \in [n]$), and relations : $v \mapsto v^i$ is linear for $i \in [n]$,

$$[v^i, w^j] = \langle v, w\rangle t_{ij} \quad \text{for} \quad i \neq j \in [n],\ v, w \in V,$$

$$\sum_{a=1}^{g} [x_a^i, y_a^i] = -\sum_{j: j\neq i} t_{ij}, \quad \forall i \in [n],$$

$$[v^i, t_{jk}] = 0 \quad \text{for} \quad i, j, k \in [n] \text{ different}, \quad v \in V.$$

Here $(V, \langle -,-\rangle)$ is a symplectic vector space of dimension $2g$, with symplectic basis $(x_a, y_a)_{a\in[g]}$ (so $\langle x_a, y_b\rangle = \delta_{ab}$). $\mathfrak{t}_{g,n}$ is equipped with a $\mathbb{N}^2$-degree given by $|x_a^i| = (1,0)$, $|y_a^i| = (0,1)$. The total degree defines a positive grading on $\mathfrak{t}_{g,n}$; we denote by $\hat{\mathfrak{t}}_{g,n}$ the corresponding completion.

Theorem 1. *([Bez]) There exists a morphism $P_{g,n} \to \exp(\hat{\mathfrak{t}}_{g,n})$, inducing an isomorphism of Lie algebras $\mathrm{Lie}(P_{g,n})^{\mathbb{C}} \stackrel{\sim}{\to} \hat{\mathfrak{t}}_{g,n}$.*

Here $\mathrm{Lie}\,\Gamma$ is the Lie algebra of the prounipotent (or Malcev) completion of a finitely generated group Γ and $V^{\mathbb{C}}$ is the complexification of a (pro-)finite dimensional $\mathbb{Q}$-vector space V.

The proof of [Bez] uses minimal model theory. The purpose of this paper is to reprove this result using explicit flat connections on configuration spaces.

2. Principal bundles and flat connections

Let X be a smooth manifold, $x \in X$, set $\Gamma := \pi_1(X, x)$. Let G_0 be a complex proalgebraic group, $\mathfrak{g}_0$ be its Lie algebra. Fix a morphism $\Gamma \stackrel{\rho_0}{\to} G_0$. It gives rise to a principal G_0-bundle $P_0 \to X$, equipped with a flat connection ∇_0.

Let U be a prounipotent complex group, equipped with an action of G_0 and $G := U \rtimes G_0$. Let $\mathfrak{u}$, $\mathfrak{g}$ be the corresponding Lie algebras, then $\mathfrak{g} = \mathfrak{u} \rtimes \mathfrak{g}_0$. These Lie algebras are equipped with decreasing filtrations $\mathfrak{u} = \mathfrak{u}^1 \supset \mathfrak{u}^2 \supset \cdots$ and $\mathfrak{g} = \mathfrak{g}^0 \supset \mathfrak{u}^1 \supset \mathfrak{u}^2 \supset \cdots$ (with the convention $[\mathfrak{x}^i, \mathfrak{x}^j] \subset \mathfrak{x}^{i+j}$).

Let $(P, \nabla) := (P_0, \nabla_0) \times_{G_0} G$ be the principal G-bundle with flat connection over X obtained by change of groups. The set of flat connections on this bundle is $\mathcal{F} = \{\alpha \in \Omega^1(X, \mathrm{ad}\,P) | d\alpha = \alpha \wedge \alpha\}$, where $\mathrm{ad}\,P = P \times_G \mathfrak{g}$. The filtration of $\mathfrak{g}$ induces a decreasing filtration $\mathrm{ad}\,P = (\mathrm{ad}\,P)^0 \supset (\mathrm{ad}\,P)^1 \supset \cdots$ and we set $\mathcal{F}^1 := \mathcal{F} \cap \Omega^1(X, (\mathrm{ad}\,P)^1)$. Then holonomy gives rise to a map $\mathcal{F}^1 \to \mathrm{Def}(\rho_0) := \{\text{lifts } \rho : \Gamma \to G \text{ of } \rho_0\}$. A lift of ρ_0 is a morphism $\Gamma \stackrel{\rho}{\to} G$ such that $(\Gamma \stackrel{\rho}{\to} G \to G_0) = (\Gamma \stackrel{\rho_0}{\to} G_0)$.

[1] We set $[n] := \{1, \ldots, n\}$.

In the particular case where $\mathfrak{u}$ is graded ($\mathfrak{u} = \hat\oplus_{i\geq 1}\mathfrak{u}_i$, where $[\mathfrak{u}_i,\mathfrak{u}_j] \subset \mathfrak{u}_{i+j}$), $(\mathrm{ad}\, P)^1$ is graded: $(\mathrm{ad}\, P)^1 = \hat\oplus_{i\geq 1}(\mathrm{ad}\, P)_i$, where $(\mathrm{ad}\, P)_i = P_0 \times_{G_0} \mathfrak{u}_i$. Then $\mathcal{F}_1 := \mathcal{F}^1 \cap \Omega^1(X, (\mathrm{ad}\, P)_1) = \{\alpha \in \Omega^1(X, (\mathrm{ad}\, P)_1) | d\alpha = \alpha \wedge \alpha = 0\}$.

We obtain in particular a map $\mathcal{F}_1 \to \mathrm{Def}(\rho_0)$. The morphism ρ associated to α expands as

$$\rho(\gamma) = \rho_0(\gamma) \exp(\int_x^{\gamma x} \alpha + (\text{element of } \mathfrak{u}^2)). \tag{1}$$

Let Σ be a finite group. Let $P_0 \to X$ be a principal bundle over a smooth manifold X with underlying group G_0. Assume that the situation is Σ-equivariant, i.e.: Σ acts by automorphisms of G_0 and X, and the action of Σ lifts to P_0 compatibly with its action on G_0. Assume that the action of Σ on X is free, and let $\tilde X := X/\Gamma$ be the smooth quotient. Then $P_0 \to X/\Gamma = \tilde X$ is a $G_0 \rtimes \Sigma$-bundle. An equivariant connection on $P_0 \to X$ induces a connection on $P_0 \to \tilde X$, and therefore a morphism $\pi_1(\tilde X) \to G_0 \rtimes \Sigma$, such that

$$\begin{array}{ccc} \pi_1(X) & \stackrel{\rho_0}{\to} & G_0 \\ \downarrow & & \downarrow \\ \pi_1(\tilde X) & \stackrel{\tilde\rho_0}{\to} & G_0 \rtimes \Sigma \end{array}$$

commutes.

The set of flat connections on $P_0 \to \tilde X$ is the set of flat equivariant connections on $P_0 \to X$, i.e., $\mathcal{F}^{eq} = \mathcal{F} \cap \Omega^1(X, \mathrm{ad}\, P_0)^\Sigma$.

Let $G = U \rtimes G_0$ as above, and assume that Σ acts compatibly on U and G_0, and therefore on G. Then $(P, \nabla) = (P_0, \nabla_0) \times_{G_0} G$ is a Σ-equivariant G-bundle over X, and therefore a $G \rtimes \Sigma$-bundle over $\tilde X = X/\Sigma$. Set $\mathcal{F}^{1,eq} := \mathcal{F}^1 \cap \mathcal{F}^{eq}$, then holonomy gives a map $\mathcal{F}^{1,eq} \to \mathrm{Def}(\rho_0, \tilde\rho_0)$, by which we understand the set of pairs $(\rho, \tilde\rho)$ lifting $(\rho_0, \tilde\rho_0)$, such that

$$\begin{array}{ccc} \pi_1(X) & \stackrel{\rho}{\to} & G \\ \downarrow & & \downarrow \\ \pi_1(\tilde X) & \stackrel{\tilde\rho}{\to} & G \rtimes \Sigma \end{array}$$

commutes.

If $\mathfrak{u}$ is Γ-equivariantly graded, then $\mathcal{F}_1^{eq} = \mathcal{F}_1 \cap \mathcal{F}^{1,eq} = \{\alpha \in \Omega^1(X, P_0 \times_{G_0} \mathfrak{u}_1)^\Sigma | d\alpha = \alpha \wedge \alpha = 0\}$. Holonomy gives a map $\mathcal{F}_1^{eq} \to \mathrm{Def}(\rho_0, \tilde\rho_0)$.

3. The main results

3.1. **The structure of some Lie algebras.** Let $g \geq 1$, $n \geq 0$ be integers.

Lemma 2. *Let $\mathfrak{u} := \oplus_{p\geq 0, q>0}\mathfrak{t}_{g,n}[p,q]$, then there is an isomorphism $\mathfrak{t}_{g,n} \simeq \mathfrak{u} \rtimes \mathfrak{f}_g^{\oplus n}$, where $\mathfrak{f}_g$ is the free Lie algebra with g generators.*

Proof. Let $(x_a)_{a\in[g]}$ be the generators of $\mathfrak{f}_g$, then there is a unique morphism $\mathfrak{f}_g^{\oplus n} \to \mathfrak{t}_{g,n}$ with $x_a^{(i)} \mapsto x_a^i$, where $x \mapsto x^{(i)}$ is the ith inclusion $\mathfrak{f}_g \to \mathfrak{f}_g^{\oplus n}$. On the other hand, the quotient $\mathfrak{t}_{g,n}/(y_a^i, a \in [g], i \in [n])$ is presented by generators x_a^i, $a \in [g], i \in [n]$ and relations $[x_a^i, x_b^j] = 0$ for $i \neq j$, hence is isomorphic to $\mathfrak{f}_g^{\oplus n}$. As the composed map $\mathfrak{f}_g^{\oplus n} \to \mathfrak{t}_{g,n} \to \mathfrak{f}_g^{\oplus n}$ is the identity, $\mathfrak{t}_{g,n} \simeq \mathrm{Ker}(\mathfrak{t}_{g,n} \to \mathfrak{f}_g^{\oplus n}) \rtimes \mathfrak{f}_g^{\oplus n}$. The result follows from $\mathrm{Ker}(\mathfrak{t}_{g,n} \to \mathfrak{f}_g^{\oplus n}) = \mathfrak{u}$. □

We set $G_0 := \exp(\hat{\mathfrak{f}}_g^{\oplus n})$ and $G := \exp(\hat{\mathfrak{t}}_{g,n})$; these groups are as in Section 2.

3.2. **Flat connections on configuration spaces and formality.** Define $\pi_g := \langle A_a, B_a, a \in [g] | \prod_{a=1}^g (A_a, B_a) = 1\rangle$.

Assume that the following data is given :

- a smooth, closed complex curve C ;
- a point $x = (x_1, \ldots, x_n) \in \mathrm{Cf}_n(C)$;

• a collection of isomorphisms $\pi_1(C, x_i) \xrightarrow{\sim} \pi_g$, such that the resulting isomophisms $\pi_1(C, x_i) \to \pi_1(C, x_j)$ are induced by a path from x_i to x_j.

We set $X := C^n - (\text{diagonals})$, $\Gamma := \pi_1(X, x)$ as in Subsection 2. Then $\Gamma \simeq P_{g,n}$.

Define $\rho_0 : P_{g,n} \to \exp(\hat{\mathfrak{f}}_g^{\oplus n}) = G_0$ as the composite map $P_{g,n} = \pi_1(\mathrm{Cf}_n(C), x) \to \pi_1(C^n, x) = \prod_{i\in[n]} \pi_1(C, x_i) \to \pi_g^n \to F_g^n \to \exp(\hat{\mathfrak{f}}_g)^n = G_0$, where F_g is the free group with generators $\gamma_a, a \in [g]$, $\pi_g \to F_g$ is the composite of the quotient morphism $\pi_g \to \pi_g/N$, where N is the normal subgroup generated by the A_a, $a \in [g]$ and $\pi_g/N \to F_g$, $\bar{B}_a \mapsto \gamma_a$ is the isomorphism arising from the presentation of π_g/N, and $F_g \to \exp(\hat{\mathfrak{f}}_g)$ is given by $\gamma_a \mapsto \exp(x_a)$.

The principal G-bundle with flat connection on $X = \mathrm{Cf}_n(C)$ corresponding to ρ_0 (analogue of (P, ∇) in Section 2) is then $i^*(\mathcal{P}_n)$, where $i : X \to C^n$ is the inclusion and $(\mathcal{P}_n \to C^n) = (\mathcal{P}_1^0 \to C)^n \times_{\exp(\hat{\mathfrak{f}}_g)^n} \exp(\hat{\mathfrak{t}}_{g,n})$, where $(\mathcal{P}_1^0 \to C)$ is the principal $\exp(\hat{\mathfrak{f}}_g)$-bundle with flat connection corresponding to the above morphism $\pi_g \to F_g \to \exp(\hat{\mathfrak{f}}_g)$.

The set of flat connections of degree 1 is then

$$\mathcal{F}_1 = \{\alpha \in \Omega^1(C^n - (\text{diagonals}), \mathcal{P}_n \times_{\mathrm{ad}} \hat{\mathfrak{t}}_{g,n}[1]) | d\alpha = \alpha \wedge \alpha = 0\}$$

and its subset of holomorphic flat connections is

$$\mathcal{F}_1^{hol} = \{\alpha \in H^0(C^n, \Omega^{1,0}_{C^n} \otimes (\mathcal{P}_n \times_{\mathrm{ad}} \hat{\mathfrak{t}}_{g,n}[1])(*\Delta)) | d\alpha = \alpha \wedge \alpha = 0\}$$

where $\Delta = \sum_{i<j} \Delta_{ij}$ and $\Delta_{ij} \subset C^n$ is the diagonal corresponding to (i, j). In Subsection 4, we will show:

Theorem 3. *A particular explicit element $\alpha_{KZ} \in \mathcal{F}_1^{hol}$ can be constructed as a sum*

$$\alpha_{KZ} = \sum_{i=1}^{n} \alpha_i, \tag{2}$$

where $\alpha_i \in H^0(C, K_C^{(i)} \otimes (\mathcal{P}_n \times_{\mathrm{ad}} \hat{\mathfrak{t}}_{g,n}[1])(\sum_{j:j\neq i} \Delta_{ij}))$ expands as $\alpha_i \equiv \sum_{a\in[g]} \omega_a^{(i)} y_a^i$ modulo $\hat{\oplus}_{q\geq 2} \mathfrak{t}_{g,n}[1, q]$.

Here $K_C^{(i)} = \mathcal{O}_C^{\boxtimes i-1} \boxtimes K_C \boxtimes \mathcal{O}_C^{\boxtimes n-i}$, $\omega_a^{(i)} = 1^{\otimes i-1} \otimes \omega_a \otimes 1^{\otimes n-i}$, where $(\omega_a)_{i\in[g]}$ are the holomorphic differentials such that $\int_{\mathcal{A}_a} \omega_b = \delta_{ab}$ and $\mathcal{A}_a, \mathcal{B}_a$ are the images of A_a, B_a under $\pi_g \to \pi_g^{ab} \simeq H_1(C, \mathbb{Z})$.

The group $P_{g,n}$ is the kernel of the morphism $B_{g,n} \to S_n$. According to [Bell], $B_{g,n}$ is presented by generators X_a, Y_a, σ_i ($a \in [g]$, $i \in [n-1]$) and relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ if } i \in [n-2], \quad (\sigma_i, \sigma_j) = 1 \text{ if } |i-j| > 1, \tag{3}$$

$$(X_a, \sigma_i) = (Y_a, \sigma_i) = 1 \text{ if } i > 1, a \in [g], \tag{4}$$

$$(\sigma_1^{-1}X_a\sigma_1^{-1}, X_a) = (\sigma_1^{-1}Y_a\sigma_1^{-1}, Y_a) = 1 \text{ if } a \in [g], \tag{5}$$

$$(\sigma_1^{-1}X_a\sigma_1^{-1}, X_b) = (\sigma_1^{-1}X_a\sigma_1^{-1}, Y_b) = (\sigma_1^{-1}Y_a\sigma_1^{-1}, X_b) = (\sigma_1^{-1}Y_a\sigma_1^{-1}, Y_b) = 1 \text{ if } a < b, \tag{6}$$

$$(\sigma_1(X_a)^{-1}\sigma_1, (Y_a)^{-1}) = \sigma_1^2 \text{ if } a \in [g], \tag{7}$$

$$\prod_{a\in[g]} (X_a, (Y_a)^{-1}) = \sigma_1 \cdots \sigma_{n-1}^2 \cdots \sigma_1. \tag{8}$$

The morphism $B_{g,n} \to S_n$ is given by $X_a, Y_a \mapsto 1$, $\sigma_i \mapsto s_i := (i, i+1)$. It is proved in [Bell] that $P_{g,n}$ is generated by X_a^i, Y_a^i ($i \in [n]$, $a \in [g]$), where $Z_a^i = \sigma_{i-1}^{-1} \cdots \sigma_1^{-1} Z_a \sigma_1^{-1} \cdots \sigma_{i-1}^{-1}$ for Z any of the letters X, Y.

One can prove that the group with the same presentation as $B_{g,n}$ together with the additional relations $\sigma_i^2 = 1$ ($i \in [n-1]$) is isomorphic to $(\pi_g)^n \rtimes S_n$. It follows that there is a natural morphism $B_{g,n} \to (\pi_g)^n \rtimes S_n$, which restricts to $P_{g,n} \to \pi_g^n$. The images of X_a^i, Y_a^i under this morphism are then $A_a^{(i)}, B_a^{(i)}$, where $\gamma \mapsto \gamma^{(i)}$ is the ith inclusion $\pi_g \to \pi_g^n$.

In view of the expansion (1), the morphism $\rho : P_{g,n} \to G = \exp(\hat{\mathfrak{t}}_{g,n})$ associated to α_{KZ} is given by $X_a^i \mapsto e^{y_a^i + \hat{\mathfrak{t}}_{g,n}^{\geq 2}}$, $Y_a^i \mapsto e^{x_a^i + \sum_b \tau_{ab} y_b^i + \hat{\mathfrak{t}}_{g,n}^{\geq 2}}$, where $\tau_{ab} = \int_{\mathcal{B}_a} \omega_b$ and $\hat{\mathfrak{t}}_{g,n}^{\geq 2} = \hat{\oplus}_{p+q\geq 2} \mathfrak{t}_{g,n}[p,q]$.

By a standard argument, we derive from Theorem 3 the formality of $P_{g,n}$.

Theorem 4. *(see also* [Bez]*) The morphism* $(\mathrm{Lie}\, P_{g,n})^{\mathbb{C}} \to \hat{\mathfrak{t}}_{g,n}$ *induced by* ρ *is an isomorphism of filtered Lie algebras.*

Proof. Recall the properties of prounipotent completion. If Γ is a finitely generated group, its prounipotent completion is a $\mathbb{Q}$-group scheme $\Gamma(-)$. There is a group morphism $\Gamma \to \Gamma(\mathbb{Q})$ universal with respect to the morphisms $\Gamma \to U(\mathbb{Q})$, where $U(-)$ is a prounipotent $\mathbb{Q}$-group scheme. In particular, ρ gives rise to a morphism $\mathrm{Lie}\,\rho : (\mathrm{Lie}\, P_{g,n})^{\mathbb{C}} \to \hat{\mathfrak{t}}_{g,n}$ and induces a morphism $\mathrm{gr}\,\mathrm{Lie}\,\rho : (\mathrm{gr}\,\mathrm{Lie}\, P_{g,n})^{\mathbb{C}} \to \mathfrak{t}_{g,n}$.

Let $\log : \Gamma \to \mathrm{Lie}\,\Gamma$ be the composed map $\Gamma \to \Gamma(\mathbb{Q}) \stackrel{\log}{\to} \mathrm{Lie}\,\Gamma(\mathbb{Q})$. $\mathrm{gr}^1(\mathrm{Lie}\, P_{g,n})^{\mathbb{C}}$ contains classes $[\log X_i^a], [\log Y_i^a]$ and $\mathrm{gr}\,\mathrm{Lie}\,\rho$ takes these elements to y_a^i, $x_a^i + \sum_b \tau_{ab} y_b^i$, which generate $\mathfrak{t}_{g,n}$, hence $\mathrm{gr}\,\mathrm{Lie}\,\rho$ is onto, hence so is $\mathrm{Lie}\,\rho$.

Lemma 5. *There is a unique morphism* $\mathfrak{t}_{g,n} \to \mathrm{gr}\,\mathrm{Lie}\, P_{g,n}$*, such that* $x_a^i \mapsto [\log X_a^i]$*,* $y_a^i \mapsto [\log Y_a^i]$*.*

Proof of Lemma. Set $\tilde{x}_a := \log X_a \in \mathrm{Lie}\, P_{g,n}$, $\tilde{y}_a := \log Y_a \in \mathrm{Lie}\, P_{g,n}$.

The morphism $B_n \to B_{g,n}$ defined by $\sigma_i \mapsto \sigma_i$ restricts to a morphism $P_n \to P_{g,n}$. The group $\mathrm{im}(B_n \times_{S_n} S_{n-1} \to B_{g,n})$ (the inclusion is $S_{n-1} \to S_1 \times S_{n-1} \to S_n$) is generated by $\mathrm{im}(P_n \to P_{g,n})$ and the σ_i, $i \geq 2$. Relations (4) then imply that for any $g \in \mathrm{im}(B_n \times_{S_n} S_{n-1} \to B_{g,n})$, $g\tilde{x}_a g^{-1} \equiv \tilde{x}_a$, $g\tilde{y}_a g^{-1} \equiv \tilde{y}_a$ modulo $F^2\,\mathrm{Lie}\, P_{g,n}$ (we set $F^1\mathfrak{g} = \mathfrak{g}$, $F^{i+1}\mathfrak{g} = [\mathfrak{g}, F^i\mathfrak{g}]$ for $\mathfrak{g}$ a Lie algebra). This implies that the classes modulo $F^2\,\mathrm{Lie}\, P_{g,n}$ of $\tau_i \tilde{x}_a \tau_i^{-1}$, $\tau_i \tilde{y}_a \tau_i^{-1}$ are independent of the choice of $\tau_i \in \mathrm{im}(B(i) \to B_{g,n})$, where $B(i) = B_n \times_{S_n} S(i)$ and $S(i) = \{\sigma \in S_n | \sigma(1) = i\}$. We denote by $\underline{x}_a^i, \underline{y}_a^i \in \mathrm{gr}_1\,\mathrm{Lie}\, P_{g,n}$ these classes.

Let $\tilde{t}_{12} := \log \sigma_1^2 \in \mathrm{Lie}\, P_{g,n}$. Relation (7) implies that $\tilde{t}_{12} \in F^2\,\mathrm{Lie}\, P_{g,n}$. We denote by $\underline{t}_{12}$ the class of $\tilde{t}_{12}$ in $\mathrm{gr}_2\,\mathrm{Lie}\, P_{g,n}$. The group $\mathrm{im}(B_n \times_{S_n} (S_2 \times S_{n-2}) \to B_{g,n})$ is generated by $\mathrm{im}(P_n \to B_{g,n})$ and $\sigma_1, \sigma_3, \ldots, \sigma_{n-1}$. Then relations (3) imply that for any $i \neq j$, the class of $\tau_{ij} \tilde{t}_{12} \tau_{ij}^{-1}$ is independent of the choice of $\tau_{ij} \in \mathrm{im}(B(i,j) \to B_{g,n})$, where $B(i,j) = B_n \times_{S_n} S(i,j)$ and $S(i,j) = \{\sigma \in S_n | \sigma(\{1,2\}) = \{i,j\}\}$. We denote by $\underline{t}_{ij} \in \mathrm{gr}_2\,\mathrm{Lie}\, P_{g,n}$ this class.

Relation (3) implies $(X_a, \sigma_2^2) = (Y_a, \sigma_2^2) = 1$ (relation in $P_{g,n}$), which yields by taking logarithms and classes modulo $F^4\,\mathrm{Lie}\, P_{g,n}$ the relations $[\underline{x}_a, \underline{t}_{23}] = [\underline{y}_a, \underline{t}_{23}] = 0$ in $\mathrm{gr}_3\,\mathrm{Lie}\, P_{g,n}$. Conjugating these relations in $P_{g,n}$ by $\tau_{ijk} \in \mathrm{im}(B(i,j,k) \to B_{g,n})$, where $B(i,j,k) = B_n \times_{S_n} S(i,j,k)$ and $S(i,j,k) = \{\sigma \in S_n | \sigma(1) = i, \sigma(2) = j, \sigma(k) = k\}$ and applying the same procedure, one obtains the relations $[\underline{x}_a^i, \underline{t}_{jk}] = [\underline{y}_a^i, \underline{t}_{jk}] = 0$.

Similarly, relations (5) imply by taking logarithms and classes modulo $F^3\,\mathrm{Lie}\, P_{g,n}$ the relations $[\underline{x}_a^1, \underline{x}_a^2] = [\underline{y}_a^1, \underline{y}_a^2] = 0$ in $\mathrm{gr}_2\,\mathrm{Lie}\, P_{g,n}$. Conjugating these relations by $\tau_{ij} \in \mathrm{im}(B(i,j) \to B_{g,n})$ and applying the same procedure, one obtains the relations $[\underline{x}_a^i, \underline{x}_a^j] = [\underline{y}_a^i, \underline{y}_a^j] = 0$ for any $i \neq j$; In the same way, relations (6) yield relations $[\underline{x}_a^i, \underline{x}_b^j] = [\underline{x}_a^i, \underline{y}_b^j] = [\underline{y}_a^i, \underline{y}_b^j] = 0$ for $a \neq b$ and $i \neq j$.

Finally, relation (7) implies by taking logarithms and classes the relations $[\underline{x}_a^2, \underline{y}_a^1] = \underline{t}_{12}$, and by conjugating beforehand by an element of $\mathrm{im}(B(j,i) \to B_{g,n})$ the relations $[\underline{x}_a^i, \underline{y}_a^j] = \underline{t}_{ij}$, and relation (8) implies $\sum_a [\underline{x}_a^i, \underline{y}_a^i] + \sum_{j : j \neq i} \underline{t}_{ij} = 0$.

All this implies that there is a unique morphism $\underline{\mathfrak{t}}_{g,n} \to \mathrm{gr}\,\mathrm{Lie}\,P_{g,n}$, such that $\underline{x}_i^a \mapsto x_a^i$, $\underline{y}_i^a \mapsto y_a^i$. $\square$

End of proof of Theorem. There is a unique automorphism $\theta \in \mathrm{Aut}(\mathfrak{t}_{g,n})$, such that $x_a^i \mapsto y_a^i$, $y_a^i \mapsto x_a^i + \sum_b \tau_{ab} y_b^i$. The composed morphism $\mathrm{gr}\,\mathrm{Lie}\,P_{g,n} \stackrel{\mathrm{gr}\,\mathrm{Lie}\,\rho}{\to} \mathfrak{t}_{g,n} \stackrel{\theta^{-1}}{\to} \mathfrak{t}_{g,n} \to \mathrm{gr}\,\mathrm{Lie}\,P_{g,n}$ takes $[\log X_a^i]$, $[\log Y_a^i]$ to themselves; as these elements generate $\mathrm{gr}\,\mathrm{Lie}\,P_{g,n}$, this is the identity. It follows that $\mathrm{gr}\,\mathrm{Lie}\,\rho$ is injective. So $\mathrm{gr}\,\mathrm{Lie}\,\rho$ is a filtered isomorphism. $\square$

Using S_n-equivariance, the holonomy morphism $P_{g,n} \to \exp(\hat{\mathfrak{t}}_{g,n})$ may be enhanced as follows.

Note that the bundle $i^*(\mathcal{P}_n) \to \mathrm{Cf}_n(C)$ is S_n-equivariant, so it gives rise to a $\exp(\hat{\mathfrak{t}}_{g,n}) \rtimes S_n$-bundle $i^*(\mathcal{P}_n) \to \mathrm{Cf}_{[n]}(C)$. The 1-form α_{KZ} is S_n-equivariant, so the monodromy representation $P_{g,n} \to \exp(\hat{\mathfrak{t}}_{g,n})$ extends to a morhism

$$\tilde\rho : B_{g,n} \to \exp(\hat{\mathfrak{t}}_{g,n}) \rtimes S_n. \tag{9}$$

The undeformed version $\tilde\rho_0$ of $\tilde\rho$ is constructed as follows. There exists a unique morphism $B_{g,n} \to \pi_g^n \rtimes S_n$, such that

$$\begin{array}{ccccc} P_n & \hookrightarrow & B_{g,n} & \hookleftarrow & P_{g,n} \\ \downarrow & & \downarrow & & \downarrow \\ S_n & \hookrightarrow & \pi_g^n \rtimes S_n & \hookleftarrow & \pi_g^n \end{array}$$

commutes. Then $(B_{g,n} \stackrel{\tilde\rho_0}{\to} \exp(\hat{\mathfrak{f}}_g)^n \rtimes S_n) = (B_{g,n} \to \pi_g^n \rtimes S_n \to F_g^n \rtimes S_n \to \exp(\hat{\mathfrak{f}}_n)^n \rtimes S_n)$.

4. The construction of α_{KZ}

4.1. **The geometric setup.** Pick x_0 in C. Fix an isomorphism $\pi_1(C, x_0) \stackrel{\sim}{\to} \pi_g$ compatible with the isomorphisms $\pi_1(C, x_i) \stackrel{\sim}{\to} \pi_g$. Let $C_{univ} \stackrel{p}{\to} C$ be the universal cover of C, then the choice of a lift of x_0 gives rise to an isomorphism $\mathrm{Aut}\,p \simeq \pi_1(C, x_0)$, and therefore to an isomorphism $\mathrm{Aut}\,p \simeq \pi_g$. Let $\tilde C := C_{univ}/N$, then $\tilde C \to C$ is a covering with group $F_g = \pi_g/N$.

There is a unique isomorphism $\pi_g \simeq \langle \tilde A_a, \tilde B_a, a \in [g] | \tilde A_1 \cdots \tilde A_g = (\tilde B_1 \tilde A_1 \tilde B_1^{-1}) \cdots (\tilde B_g \tilde A_g \tilde B_g^{-1})\rangle$, given by

$$\tilde A_a = (\prod_{b<a} B_b A_b^{-1} B_b^{-1}) \cdot A_a \cdot (\prod_{b<a} B_b A_b^{-1} B_b^{-1})^{-1}, \quad \tilde B_a = (\prod_{b<a} B_b A_b^{-1} B_b^{-1}) \cdot B_a \cdot (\prod_{b<a} B_b A_b^{-1} B_b^{-1})^{-1}.$$

Cut out on C and with homotopy classes $\tilde B_1$, $\tilde A_1$, $\tilde B_1^{-1}$, $\ldots$, $\tilde B_g$, $\tilde A_g$, $\tilde B_g^{-1}$, $\tilde A_g^{-1}, \ldots, \tilde A_1^{-1}$. The lifts of these loops to $\tilde C$ are a collection of successive paths p_1, $\mathcal{A}_1$, p_1^{-1}, $\ldots$, p_g, $\mathcal{A}_g$, p_g^{-1}, $\gamma_1^{-1}(\mathcal{A}_1)^{-1}$, $\ldots$, $\gamma_g^{-1}(\mathcal{A}_g)^{-1}$. They cut out a fundamental domain $\tilde D \subset \tilde C$, such that $\partial \tilde D = \cup_{a\in[g]} \mathcal{A}_a \cup \gamma_a^{-1}(\mathcal{A}_a)$.

The residue formula is then

$$\sum_{P \in \tilde D} \mathrm{res}_P(\omega) + \sum_{a\in[g]} \int_{\mathcal{A}_a} (\gamma_a - 1)(\omega) = 0$$

for ω any meromorphic differential on $\tilde C$.

4.2. **Conditions on α_i and its properties.** Let $\mathbf{z} = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n) \in \tilde C^{n-1} \times_{C^{n-1}} \mathrm{Cf}_{n-1}(C)$. Let $\mathbf{z}$ denote also the divisor $z_1 + \cdots + z_n$ of $\tilde C$.

Lemma 6. *There exists a unique $\alpha_i^{\mathbf{z}} \in H^0(\tilde C, K_C(\mathbf{z})) \otimes \hat{\mathfrak{t}}_{g,n}[1]$, such that*

- $\forall a \in [g], \gamma_a(\alpha_i^{\mathbf{z}}) = e^{\mathrm{ad}\, x_a^i}(\alpha_i^{\mathbf{z}})$,
- $\forall j \neq i, \mathrm{res}_{z_j}(\alpha_i^{\mathbf{z}}) = t_{ij}$,
- $\int_{\mathcal{A}_a} \alpha_i^{\mathbf{z}} = \frac{\mathrm{ad}\, x_a^i}{e^{\mathrm{ad}\, x_a^i} - 1}(y_a^i)$.

Let $\tilde\Delta_i$ be the divisor of $\tilde C^n$, preimage of $\Delta_i = \Delta_{i1} + \cdots + \Delta_{in}$ under $p : \tilde C^n \to C^n$.

There exists a unique $\alpha_i \in H^0(\tilde C^n, K_{\tilde C}^{(i)}(\tilde\Delta_i))\hat\otimes\hat{\mathfrak t}_{g,n}$, such that $(\alpha_i)_{|(z_1,\ldots,z_{i-1})\times\tilde C\times(z_{i+1},\ldots,z_n)} = \alpha_i^{\mathbf z}$.

Proposition 7. *For $i\in[n]$ and $a\in[g]$, $\gamma_a^j(\alpha_i) = e^{\mathrm{ad}\, x_a^j}(\alpha_i)$, so that $\alpha_i \in H^0(C^n, K_C^{(i)} \otimes \mathrm{ad}\,\mathcal P_n(\Delta_i))$. One also has $\mathrm{res}_{ij}(\alpha_i) = t_{ij}$.*

For X a variety and $\mathcal E \to C\times C\times X$ a bundle, the residue is a map $H^0(C\times C\times X, (K_C \boxtimes \mathcal O_C(*\Delta)\boxtimes\mathcal O_X)\otimes\mathcal E) \to H^0(C\times X, (p\times \mathrm{id}_X)^*(\mathcal E))$, where $p : C\to C\times C$ is the diagonal map and $\Delta\subset C\times C$ is the diagonal divisor. One similarly defines $\mathrm{res}_{ij} : H^0(C^n, K_C^{(i)}\otimes\mathcal E(*\Delta_{ij})) \to H^0(C^{n-1}, p_{ij}^*(\mathcal E))$, where $p_{ij} : C^{n-1}\to C^n$ is the composition with the map $[n]\to[n-1]$, inducing an increasing bijection $[n]-\{i,j\}\to[n-1]-\{1\}$ and such that $i,j\mapsto 1$.

4.3. **Geometric material.** An element $\alpha\in H^0(\tilde C^n, K_{\tilde C}^{(i)}(\Delta_i))$ will be denoted $\alpha(z_1,\ldots,z_n)dz_i = \alpha^{z_1\cdots\underline{z_i}\cdots z_n}$. The action of $\gamma\in F_g$ on this space, induced by its action on the jth component of $\tilde C^n$ is denoted by $\gamma^j = \gamma^{(z_j)}$. When $n=2$, one sets $(z_1,z_2)=(z,w)$.

Lemma 8. *There is a unique family $\omega_{a_1\ldots a_s}^{\underline{z}w} \in H^0(\tilde C\times\tilde C, K_{\tilde C}\boxtimes\mathcal O_{\tilde C}(\tilde\Delta))$, where $s\geq 1, (a_1,\ldots,a_s)\in[g]^s$, such that:*

- *for $n=1$, $\omega_a^{\underline{z}w} = \omega_a^{\underline z}$;*
-
$$\gamma_a^{(z)}(\omega_{a_1\ldots a_s}^{\underline{z}w}) = \sum_{k\geq 0}\frac{1}{k!}\delta_{aa_1\ldots a_k}\omega_{a_{k+1}\ldots a_s}^{\underline{z}w},$$
- $\mathrm{res}_{z=w}(\omega_{a_1\ldots a_s}^{\underline{z}w}) = -\delta_{s2}\delta_{a_1a_2}$.

Proof of Lemma. By the residue formula, the conditions on $\omega_{a_1\ldots a_s}^{\underline{z}w}$ are

$$(\gamma_a^{(z)}-1)\omega_{a_1\ldots a_s}^{\underline{z}w} = \sum_{k\geq 1}\frac{1}{k!}\delta_{aa_1\ldots a_k}\omega_{a_{k+1}\ldots a_s}^{\underline{z}w}, \qquad \int_{\mathcal A_a}^z \omega_{a_1\ldots a_s}^{\underline{z}w} = b_s\delta_{aa_1\ldots a_s},$$

where $\sum_{k\geq 1} b_k t^{k-1} = t/(e^t-1)$. Assume that the $\omega_{a_1\ldots a_t}^{\underline{z}w}$ are determined for $t<s$ and let us show that this condition determines the $\omega_{a_1\ldots a_s}^{\underline{z}w}$ uniquely.

The uniqueness of $\omega_{a_1\ldots a_s}^{\underline{z}w}$ satisfying these conditions is clear. Let us prove their existence.

Define a vector bundle $\mathcal L_s$ over C inductively by $\mathcal L_0 = K_C$,

$$\Gamma(U,\mathcal L_s) = \{\omega\in\Gamma(\tilde U, K_{\tilde C}) | \exists(\alpha_a)_{a\in[g]}\in\Gamma(U,\mathcal L_{s-1})^g, \text{ s.t. } \forall a\in[g], (\gamma_a-1)\omega = \alpha_a\},$$

where for any open subset $U\subset C$, $\tilde U := \tilde C\times_C U$. It fits in an exact sequence $0\to K_C\to\mathcal L_s\to\mathcal L_{s-1}^{\oplus g}\to 0$. For each point $\bar w\in C$, it gives rise to the exact sequence $H^0(C,\mathcal L_s(\bar w))\to H^0(C,\mathcal L_{s-1}(\bar w))^g \to H^1(C, K_C(\bar w))$. By Serre duality, $H^1(C,K_C(\bar w))=0$, which implies the surjectivity of the first map, hence the existence of the $\omega_{a_1\ldots a_s}^{\underline{z}w}$. One then proves easily that the $\omega_{a_1\ldots a_s}^{\underline{z}w}$ depend meromorphically on w. □

Lemma 9. *([Fay], Cor. 2.6) There exists a unique $\psi^{\underline{zw}}\in H^0(C\times C, K_C^{\boxtimes 2}(2\Delta))$, such that:*

- *$\psi^{\underline{zw}}$ expands as $d_zd_w\log(z-w)+O(1)$ at the vicinity of the diagonal;*
- *$\int_{\mathcal A_a}^z\psi^{\underline{zw}} = 0$.*

$\psi^{\underline{zw}}$ is called the basic bidifferential in the theory of complex curves.

Lemma 10. *There is a unique family $\psi_{a_1\ldots a_s}^{\underline{zw}}\in H^0(\tilde C\times\tilde C, K_{\tilde C}^{\boxtimes 2}(2\tilde\Delta))$, where $s\geq 0, (a_1,\ldots,a_s)\in[g]^s$, such that:*

- *if $s=0$, then $\psi_{a_1\ldots a_s}^{\underline{zw}} = \psi^{\underline{zw}}$,*

•
$$\gamma_a^{(z)}(\psi^{\underline{zw}}_{a_1\ldots a_s})=\sum_{k\geq 0}\frac{1}{k!}\delta_{aa_1\ldots a_k}\psi^{\underline{zw}}_{a_{k+1}\ldots a_s},$$

- $\int_{\mathcal{A}_a}^{z}\psi^{\underline{zw}}_{a_1\ldots a_s}=0$.
- $\psi^{\underline{zw}}_{a_1\ldots a_s}$ *is regular at the diagonal of* $\tilde C\times\tilde C$ *if* $s\geq 1$.

It satisfies the identity
$$\psi^{\underline{wz}}_{a_1\ldots a_s}=(-1)^s\psi^{\underline{zw}}_{a_s\ldots a_1}.$$

Proof. The uniqueness of the family $(\psi^{\underline{zw}}_{a_1\ldots a_s})$ is clear. As for existence, it suffices to set $\psi^{\underline{zw}}_{a_1\ldots a_s}=-d_w(\omega^{\underline{zw}}_{a_1\ldots a_s bb})$ for any $b\in[g]$.

The identity $\psi^{\underline{wz}}_{a_1\ldots a_s}=(-1)^s\psi^{\underline{zw}}_{a_s\ldots a_1}$ can be proved as follows. When $\tilde C=\mathbb{P}^1-\{\alpha_a,\beta_a,a\in[g]\}$ and γ_a are defined by $\frac{\gamma_a(z)-\alpha_a}{\gamma_a(z)-\beta_a}=q_a\frac{z-\alpha_a}{z-\beta_a}$, where $(q_a)_{a\in[g]}$ are formal variables, $\psi^{\underline{zw}}_{a_1\ldots a_s}=\sum_{\gamma\in F_g}f_{a_1\ldots a_s}(\gamma)\gamma^{(z)}d_zd_w\log(z-w)$, where
$$f_{a_1\ldots a_s}(\gamma_{e_1}^{\lambda_1}\cdots\gamma_{e_t}^{\lambda_t})=\sum_{s_1+\cdots+s_t=s}\frac{(-\lambda_1)^{s_1}}{s_1!}\cdots\frac{(-\lambda_t)^{s_t}}{s_t!}\delta_{e_1a_1\ldots a_{s_1}}\cdots\delta_{e_ta_{s_1+\cdots+s_{t-1}}\ldots a_s}.$$
So $f_{a_s\cdots a_1}(\gamma^{-1})=(-1)^sf_{a_1\cdots a_s}(\gamma)$, and since $\gamma^{(z)}d_zd_w\log(z-w)=(\gamma^{-1})^{(w)}d_zd_w\log(z-w)$, it follows that
$$\psi^{\underline{wz}}_{a_1\ldots a_s}=(-1)^s\psi^{\underline{zw}}_{a_s\ldots a_1}.$$
This identity holds on the set of Mumford curves, which is a formal neighborhood of the locus of totally degenerate curves in the moduli space of triples $(C,x_0,\pi_1(C,x_0)\tilde{\to}\pi_g)$, so it holds on the whole moduli space. □

Define $\psi^{\underline{zww'}}_{a_1\ldots a_s}\in H^0(\tilde C^3,K^{(1)}_{\tilde C}(\tilde\Delta_{12}+\tilde\Delta_{13}))$ by $\psi^{\underline{zww'}}_{a_1\ldots a_s}=\int_w^{w'}\psi^{\underline{zw''}}_{a_1\ldots a_s}$, where the integration is on the second variable. This is well-defined because $\int_{\mathcal{A}_a}^{w}\psi^{\underline{zw}}_{a_1\ldots a_s}=0$. Then the identity $\psi^{\underline{zww'}}_{a_1\ldots a_s}+\psi^{\underline{zw'w''}}_{a_1\ldots a_s}=\psi^{\underline{zww''}}_{a_1\ldots a_s}$ holds.

Lemma 11. *a) If $a_{s-1}\neq a_s$, then $\omega^{\underline{zw}}_{a_1\ldots a_s}$ is constant in the second variable, hence arises from an element of $H^0(\tilde C,K_{\tilde C})$.*

b)
$$(\gamma_a^{(w)}-1)\omega^{\underline{zw}}_{a_1\ldots a_s bb}=\sum_{k\geq 0}\frac{(-1)^{k+1}}{(k+1)!}\delta_{aa_s\ldots a_{s-k+1}}\omega^{\underline{zw}}_{a_1\ldots a_{s-k}a}\tag{10}$$

Proof. One proves inductively on s that $\omega^{\underline{zw}}_{a_1\ldots a_s}-\omega^{\underline{zw'}}_{a_1\ldots a_s}=0$. Indeed, if this is true for all indices $t<s$, then this difference satisfies $(\gamma_a^{(z)}-1)\alpha^{\underline{z}}=0$, $\int_{\mathcal{A}_a}^{z}\alpha_{\underline{z}}=0$, which implies $\alpha^{\underline{z}}=0$. This proves a).

Let us prove (10). The identities $\psi^{\underline{wz}}_{a_s\ldots a_1}=(-1)^s\psi^{\underline{zw}}_{a_1\ldots a_s}$ and
$$\gamma_a^{(z)}\psi^{\underline{zw}}_{a_1\ldots a_s}=\sum_{k\geq 0}\frac{1}{k!}\delta_{aa_1\ldots a_k}\psi^{\underline{zw}}_{a_{k+1}\ldots a_s}$$
imply $(\gamma_a^{(w)}-1)\psi^{\underline{zw}}_{a_1\ldots a_s}=\sum_{k\geq 0}\frac{(-1)^{k+1}}{(k+1)!}\delta_{aa_s\cdots a_{s-k}}\psi^{\underline{zw}}_{a_1\ldots a_{s-k-1}}$, so the images of both sides of (10) under d_w coincide. Assume that (10) has been proved at all orders $t<s$ and consider this identity at order s. As $d_w(\omega^{\underline{zw}}_{a_1\ldots a_s bb}-\omega^{\underline{zw}}_{a_1\ldots a_s cc})=\psi^{\underline{zw}}_{a_1\ldots a_s}-\psi^{\underline{zw}}_{a_1\ldots a_s}=0$, $\omega^{\underline{zw}}_{a_1\ldots a_s bb}-\omega^{\underline{zw}}_{a_1\ldots a_s cc}$ is independent of w, so (l.h.s. − r.h.s. of (10)) is a differential in z depending on $a,a_1,\ldots,a_s$ only, which we denote $\delta^{\underline{z}}_{aa_1\ldots a_s}$. Applying $\gamma_e^{(z)}-1$ to both sides of (10) and using the induction hypothesis, one obtains $(\gamma_e^{(z)}-1)\delta^{\underline{z}}_{aa_1\ldots a_s}=0$ for $e\in[g]$. The differential $\delta^{\underline{z}}_{aa_1\ldots a_s}$ is necessarily regular, as it is regular on $C-\{w\}$ for any point w, so it belongs to $H^0(C,K_C)$. To compute

it, it suffices to evaluate the integrals of both sides of (10) on a-cycles. When $s \geq 1$, $\omega^{zw}_{a_1\ldots a_s bb}$ is regular at $z = w$, so $\int_{\mathcal{A}_c}^z$(l.h.s. of (10)) $= 0$. On the other hand, $\int_{\mathcal{A}_c}^z$(r.h.s. of (10)) $= \delta_{aa_1\ldots a_s c}\sum_{k\geq 0}\frac{(-1)^{k+1}}{(k+1)!}b_{s-k+1} = 0$. So $\delta^{z}_{aa_1\ldots a_s} = 0$ for $s \geq 1$. A similar computation yields the same result for $s = 0$. □

Proposition 12.

$$\omega^{zw}_{a_1\ldots a_s bb} - \omega^{zw'}_{a_1\ldots a_s bb} = \psi^{zww'}_{a_1\ldots a_s},$$

$$\gamma_a^{(z)}(\psi^{zww'}_{a_1\ldots a_s}) = \sum_{k\geq 0}\frac{1}{k!}\delta_{aa_1\ldots a_k}\psi^{zww'}_{a_{k+1}\ldots a_n},$$

$$\gamma_a^{(w')}(\psi^{zww'}_{a_1\ldots a_s}) = \sum_{k\geq 0}\frac{(-1)^k}{k!}\delta_{aa_s\ldots a_{s-k+1}}\psi^{zww'}_{a_1\ldots a_{s-k}} + \sum_{k\geq 1}\frac{(-1)^{k-1}}{k!}\delta_{aa_s\ldots a_{s-k+2}}\omega^{zw}_{a_1\ldots a_{s-k+1}a},$$

where $\delta_{u_1\ldots u_t}$ is Kronecker's delta (= 1 by convention if $t = 1$).

Proof. The first identity follows from $\psi^{zw}_{a_1\ldots a_s} = -d_w(\omega^{zw}_{a_1\ldots a_s bb})$ by integration. The second identity follows from $\gamma_a^{(z)}\psi^{zw}_{a_1\ldots a_s} = \sum_{k\geq 0}\frac{1}{k!}\delta_{aa_1\ldots a_k}\psi^{zw}_{a_{k+1}\ldots a_s}$ by integration. Let us prove the third identity. One checks that d_w(l.h.s. − r.h.s.) $= d_{w'}$(l.h.s. − r.h.s.) $= 0$, so (l.h.s. − r.h.s.) depends on z only. Moreover, l.h.s. $= \gamma_a^{(w')}(\omega^{zw}_{a_1\ldots a_s bb} - \omega^{zw'}_{a_1\ldots a_s bb})$, while (second sum of r.h.s.) $= -(\gamma_a^{(w)} - 1)\omega^{zw}_{a_1\ldots a_s bb}$. It follows that

$$(\text{l.h.s.} - \text{r.h.s.}) = \gamma_a^{(w)}\omega^{zw}_{a_1\ldots a_s bb} - \gamma_a^{(w')}\omega^{zw'}_{a_1\ldots a_s bb} - \sum_{k\geq 0}\frac{(-1)^k}{k!}\delta_{aa_s\ldots a_{s-k+1}}\psi^{zww'}_{a_1\ldots a_{s-k}}$$

is antisymmetric in w, w'. All this implies that (l.h.s. − r.h.s.) $= 0$. □

4.4. **Construction and properties of α_i.** Set

$$\alpha_i^{z_1\ldots\underline{z_i}\ldots z_n} := \sum_{\substack{s\geq 0,\\ (a_1,\ldots,a_s,b)\in[g]^{s+1}}} \omega^{z_i w}_{a_1\ldots a_s b}[x^i_{a_1},\cdots,[x^i_{a_s},y^i_b]] + \sum_{j:j\neq i}\sum_{\substack{s\geq 0,\\ (a_1,\ldots,a_s)\in[g]^s}} \psi^{z_i w z_j}_{a_1\ldots a_s}[x^i_{a_1},\cdots,[x^i_{a_s},t_{ij}]].$$

It follows from the first identity of Proposition 12 that the r.h.s. is independent on w, which justifies the chosen notation.

Proposition 7 then follows from the identities of Proposition 12, together with the identity $[x^i_a + x^j_a, t_{ij}] = 0$ (see Lemma 18).

5. Simplicial behavior of α_{KZ}

Let $\mathcal{G} \subset \hat{\mathfrak{t}}_{g,n}$ be the Lie subalgebra generated by the $v^1 + v^2$, v^k, $k \geq 3$, $v \in V$. Then $t_{12} \in Z(\mathcal{G})$. One checks using the presentation of $\mathfrak{t}_{g,n-1}$ that there is a unique Lie algebra morphism $\mathfrak{t}_{g,n-1} \to \mathcal{G}/\mathbb{C}t_{12}$, $x \mapsto x^{12,3,\ldots,n}$, such that for $v \in V$, $(v^1)^{12,3,\ldots,n} = v^1 + v^2$, $(v^k)^{12,3,\ldots,n} = v^{k+1}$ for $k \geq 2$. In particular, $(t_{1k})^{12,\ldots,n} = t_{1,k+1} + t_{2,k+1}$, $(t_{kl})^{12,\ldots,n} = t_{k+1,l+1}$ for $k, l > 1$. We denote the same way the composed linear map $\mathfrak{t}_{g,n-1} \to \mathcal{G}/\mathbb{C}t_{12} \to \mathfrak{t}_{g,n}/\mathbb{C}t_{12}$.

When the number of marked points is $n-1$, $\alpha_1^{(n-1)}$ identifies with a differential $\alpha_1^{(n-1)} \in H^0(\tilde{C}^{n-1}, K^{(1)}_{\tilde{C}}(\Delta_{12} + \cdots + \Delta_{1,n-1})) \otimes \hat{\mathfrak{t}}_{g,n-1}$. Applying the above linear map, one gets a differential

$$(\alpha_1^{(n-1)})^{12,3,\ldots,n} \in H^0(\tilde{C}^{n-1}, K^{(1)}_{\tilde{C}}(\tilde{\Delta}_{12} + \cdots + \tilde{\Delta}_{1,n-1})) \otimes (\hat{\mathfrak{t}}_{g,n}/\mathbb{C}t_{12}).$$

If ω is a rational differential on C, let $\omega_i := 1^{\otimes i-1}\otimes\omega\otimes 1^{\otimes n-i}$ be the induced rational section of $K_C^{(i)}$ on C^n.

Let $p_{12} : C^{n-1} \to C^n$ be $(z_1, \dots , z_{n-1}) \mapsto (z_1, z_1, z_2, \dots , z_{n-1})$. Then $\Delta_{12} \subset C^n$ is the image of p_{12}.

If ω is nonzero, then as the behavior of $\alpha_i = \alpha_i^{(n)}$ $(i = 1, 2)$ on Δ_{12} is $\alpha_i = t_{12} d_{z_i} \log(z_i - z_j)$ + regular (with $\{i, j\} = \{1, 2\}$), $\frac{1}{\omega_1}(\omega_1\alpha_2 + \omega_2\alpha_1)$ is regular at Δ_{12}. We set

$$\tilde\alpha_\omega = \frac{1}{\omega_1}(\omega_1\alpha_2 + \omega_2\alpha_1)_{|\Delta_{12}},$$

which may be viewed as an element of $\Gamma_{rat}(\tilde C^{n-1}, K^{(1)}_{\tilde C}) \otimes \hat{\mathfrak t}_{g,n}$ (where Γ_{rat} means rational sections).

$\tilde\alpha_\omega$ satisfies the identity $\tilde\alpha_{f\omega} = \tilde\alpha_\omega - (d \log f)_1 t_{12}$, which implies that the class of $\tilde\alpha_\omega$ modulo $\mathbb{C}t_{12}$ satisfies

$$[\tilde\alpha_\omega] \in H^0(\tilde C^{n-1}, K^{(1)}_{\tilde C} \otimes (\tilde\Delta_{12} + \dots + \tilde\Delta_{1,n-1})) \otimes (\hat{\mathfrak t}_{g,n}/\mathbb{C}t_{12})$$

(as ω can be chosen regular at any point of C), and that this class is independent of ω.

We will prove:

Proposition 13. $(\alpha_1^{(n-1)})^{12,3,\dots,n} = [\tilde\alpha_\omega]$.

Proof. Denote the two sides by u_i, $i = 1, 2$. They have the same automorphy properties, namely $\gamma_1^a(u_i) = e^{\mathrm{ad}(x_a^1+x_a^2)}(u_i)$, $\gamma_k^a(u_i) = e^{\mathrm{ad}\, x_a^{k+1}}(u_i)$ for $k \geq 2$. They have the same poles, $\mathrm{res}_{\Delta_{1k}}\, u_i = t_{1,k+1} + t_{2,k+1}$ for $k \geq 2$. For $\mathbf{z} \in \tilde D^{n-2} \subset \tilde C^{n-2}$, we restrict the two sides to $\tilde C \times \{\mathbf{z}\}$ and show that the resulting forms $\alpha_i^{\mathbf{z}}$ have the same integrals along a-cycles.

Lemma 14. *If k is even or 1, then* $(\mathrm{ad}\, x_a^1)^k(y_a^1)^{12,3,\dots,n} = (\mathrm{ad}\, x_a^1)^k(y_a^1) + (\mathrm{ad}\, x_a^2)^k(y_a^2)$.

Proof of Lemma.

$$\begin{aligned}
&(\mathrm{ad}\, x_a^1)^k(y_a^1)^{12,3,\dots,n} = (\mathrm{ad}\, x_a^1)^k(y_a^1) + (\mathrm{ad}\, x_a^2)^k(y_a^2)\\
&+ \sum_{l=0}^{k-1} (\mathrm{ad}(x_a^1 + x_a^2))^{k-1-l}(\mathrm{ad}\, x_a^2)(\mathrm{ad}\, x_a^1)^l(y_a^1) + (\mathrm{ad}(x_a^1 + x_a^2))^{k-1-l}(\mathrm{ad}\, x_a^1)(\mathrm{ad}\, x_a^2)^l(y_a^2)\\
&= (\mathrm{ad}\, x_a^1)^k(y_a^1) + (\mathrm{ad}\, x_a^2)^k(y_a^2)\\
&+ \sum_{l=0}^{k-1} (\mathrm{ad}(x_a^1 + x_a^2))^{k-1-l}(\mathrm{ad}\, x_a^1)^l(t_{12}) + (\mathrm{ad}(x_a^1 + x_a^2))^{k-1-l}(\mathrm{ad}\, x_a^2)^l(t_{12}).
\end{aligned}$$

If $s > 0$, then $(\mathrm{ad}(x_a^1+x_a^2))^s(\mathrm{ad}\, x_a^i)^l(t_{12}) = (\mathrm{ad}\, x_a^i)^l(\mathrm{ad}(x_a^1+x_a^2))^s(t_{12}) = 0$ as $[x_a^1+x_a^2, t_{12}] = 0$. So $(\mathrm{ad}\, x_a^1)^k(y_a^1)^{12,3,\dots,n} = (\mathrm{ad}\, x_a^1)^k(y_a^1)+(\mathrm{ad}\, x_a^2)^k(y_a^2)+(\mathrm{ad}\, x_a^1)^{k-1}(t_{12})+(\mathrm{ad}\, x_a^2)^{k-1}(t_{12})$. When k is even, the sum of the two last terms vanishes.

When $k = 1$, $[x_a^1, y_a^1]^{12,3,\dots,n} = [x_a^1, y_a^1] + [x_a^2, y_a^2] + 2t_{12}$ as $[x_a^1, y_a^2] = [x_a^2, y_a^1] = t_{12}$, so $[x_a^1, y_a^1]^{12,3,\dots,n} = [x_a^1, y_a^1] + [x_a^2, y_a^2]$ as $\mathbb{C}t_{12}$ is factored out. □

There is an expansion $\frac{t}{e^t-1} = \sum_{k\in 2\mathbb{N}\cup\{1\}} b_k t^k$, so

$$\int_{\mathcal{A}_a} \alpha_1^{(n-1),\mathbf{z}} = \frac{\mathrm{ad}\, x_a^1}{e^{\mathrm{ad}\, x_a^1} - 1}(y_a^1) = \sum_{k\in 2\mathbb{N}\cup\{1\}} b_k(\mathrm{ad}\, x_a^1)^k(y_a^1).$$

Then $\int_{\mathcal{A}_a} u_1^{\mathbf{z}} = (\int_{\mathcal{A}_a} \alpha_1^{(n-1),\mathbf{z}})^{12,3,\dots,n} = \sum_{k\in 2\mathbb{N}\cup\{1\}} b_k((\mathrm{ad}\, x_a^1)^k(y_a^1) + (\mathrm{ad}\, x_a^2)^k(y_a^2))$ by Lemma 14, so

$$\int_{\mathcal{A}_a} u_1^{\mathbf{z}} = \frac{\mathrm{ad}\, x_a^1}{e^{\mathrm{ad}\, x_a^1} - 1}(y_a^1) + \frac{\mathrm{ad}\, x_a^2}{e^{\mathrm{ad}\, x_a^2} - 1}(y_a^2). \tag{11}$$

On the other hand,

$$[\tilde{\alpha}_\omega]^{\underline{z}_1,z_2,\cdots,z_{n-1}} = \sum_{s\geq 0,(a_1,\dots,a_s,b)\in[g]^{s+1}} \omega^{\underline{z}_1 w}_{a_1\dots a_s b}([x^1_{a_1},\cdots,[x^1_{a_s},y^1_b]]+[x^2_{a_1},\cdots,[x^2_{a_s},y^2_b]])$$
$$+\sum_{k=2}^{n-1}\sum_{s\geq 0,(a_1,\dots,a_s)\in[g]^s} \psi^{\underline{z}_1 w z_k}_{a_1,\dots,a_n}([x^1_{a_1},\cdots,[x^1_{a_s},t_{1,k+1}]]+[x^2_{a_1},\cdots,[x^2_{a_s},t_{2,k+1}]])$$
$$+\sum_{s\geq 1,(a_1,\dots,a_s)\in[g]^s} \psi^{\underline{z}_1 w z_1}_{a_1,\dots,a_n}([x^1_{a_1},\cdots,[x^1_{a_s},t_{12}]]+[x^2_{a_1},\cdots,[x^2_{a_s},t_{12}]]) \tag{12}$$

for any $w\in\tilde{C}$. Then:
- $\int^{z_1}_{\mathcal{A}_a}\omega^{\underline{z}_1 w}_{a_1\dots a_s b} = b_s\delta_{a,a_1,\dots,a_s,b}$ where $\sum_{s\geq 0} b_s t^s = t/(e^t-1)$;
- $\int^{z_1}_{\mathcal{A}_a}\psi^{\underline{z}_1 w z_k}_{a_1,\dots,a_s}=0$ as $\int^{z}_{\mathcal{A}_a}\psi^{\underline{zw}}_{a_1,\dots,a_s}=0$;
- $\int^{z}_{\mathcal{A}_a}\psi^{\underline{z}wz}_{a_1,\dots,a_s}$ is independent on w as $\psi^{\underline{z}wz}_{a_1,\dots,a_s}=\psi^{\underline{z}w'z}_{a_1,\dots,a_s}+\psi^{\underline{z}ww'}_{a_1,\dots,a_s}$ and $\int^{z}_{\mathcal{A}_a}\psi^{\underline{zw}}_{a_1,\dots,a_n}=0$.

To compute this integral, we assume that w lies on $\mathcal{A}_a$ and that the loop $\mathcal{A}_a$ is parametrized by $\gamma:[0,1]\to\tilde{C}$, with $\gamma(0)=\gamma(1)=w$. Then the integral under consideration appears as an iterated integral

$$\int^{z}_{\mathcal{A}_a}\psi^{\underline{z}wz}_{a_1,\dots,a_s} = -\int_{0<t_2<t_1<1}(\gamma\times\gamma)^*\psi^{\underline{z}_1\underline{z}_2}_{a_1,\dots,a_s}.$$

Using $[x^2_{a_s},\cdots,[x^2_{a_1},t_{12}]]=(-1)^s[x^1_{a_1},\cdots,[x^1_{a_s},t_{12}]]$, the contribution of the last line of (12) is

$$-\sum_{\substack{s\geq 1,\\(a_1,\dots,a_s)\in[g]^s}}\Big(\int_{0<t_2<t_1<1}(\gamma\times\gamma)^*\psi^{\underline{z}_1\underline{z}_2}_{a_1,\dots,a_s}+(-1)^s\int_{0<t_2<t_1<1}(\gamma\times\gamma)^*\psi^{\underline{z}_1\underline{z}_2}_{a_s,\dots,a_1}\Big)[x^1_{a_1},\cdots,[x^1_{a_s},t_{12}]]$$

which, taking into account $\psi^{\underline{zw}}_{a_s\dots a_1}=(-1)^s\psi^{\underline{wz}}_{a_1\dots a_s}$, is equal to

$$-\sum_{s\geq 1,(a_1,\dots,a_s)\in[g]^s}\Big(\int_{[0,1]\times[0,1]}(\gamma\times\gamma)^*\psi^{\underline{z}_1\underline{z}_2}_{a_1,\dots,a_s}\Big)[x^1_{a_1},\cdots,[x^1_{a_s},t_{12}]]$$

which vanishes as $\int^{z}_{\mathcal{A}_a}\psi^{\underline{zw}}_{a_1\dots a_s}=0$.

All this implies that

$$\int_{\mathcal{A}_a}\alpha^{\underline{z}}_2 = \frac{\operatorname{ad}x^1_a}{e^{\operatorname{ad}x^1_a}-1}(y^1_a)+\frac{\operatorname{ad}x^2_a}{e^{\operatorname{ad}x^2_a}-1}(y^2_a),$$

which, when compared with (11), ends the proof of $u_1=u_2$. □

6. The flatness of α_{KZ}

Lemma 15. $d_{z_j}\alpha_i^{z_1\cdots\underline{z}_i\cdots z_n}=d_{z_i}\alpha_j^{z_1\cdots\underline{z}_j\cdots z_n}$.

Proof.

$$d_{z_j}\alpha_i^{z_1\cdots\underline{z}_i\cdots z_n}=\sum_{s\geq 0,(a_1,\dots,a_s)\in[g]^s}\psi^{\underline{z}_i\underline{z}_j}_{a_1\dots a_s}[x^i_{a_1},\dots,[x^i_{a_s},t_{ij}]]$$
$$=\sum_{s\geq 0,(a_1,\dots,a_s)\in[g]^s}(-1)^s\psi^{\underline{z}_j\underline{z}_i}_{a_s\dots a_1}(-1)^s[x^j_{a_s},\dots,[x^j_{a_1},t_{ij}]]=d_{z_j}\alpha_i^{z_1\cdots\underline{z}_i\cdots z_n}.$$

□

Proposition 16. $[\alpha_i^{z_1\cdots\underline{z}_i\cdots z_n},\alpha_j^{z_1\cdots\underline{z}_j\cdots z_n}]=0$.

Proof. $[\alpha_i,\alpha_j]\in H^0(C^n,\mathrm{ad}\,\mathcal{P}_n\otimes K_C^{(i)}\otimes K_C^{(j)}(2\Delta_{ij}+\sum_{k\neq i,j}(\Delta_{ik}+\Delta_{jk})))$.

Let us show that $[\alpha_i,\alpha_j]$ is regular at each diagonal Δ_{ik} $(k\neq i,j)$. This quantity has a simple pole at this diagonal, with residue $[t_{ik},(\alpha_j)_{|\Delta_{ik}}]$. The form $(\alpha_j)_{|\Delta_{ik}}$ is a linear combination of (i) the $[x^j_{a_1},\cdots,[x^j_{a_s},y^j_b]]$, where $a_1,\ldots,a_s,b\in[g]$; (ii) the $[x^j_{a_1},\cdots,[x^j_{a_s},t_{jl}]]$, where $a_1,\ldots,a_s\in[g]$, $l\neq i,j,k$; (iii) the $[x^j_{a_1},\cdots,[x^j_{a_s},t_{ji}+t_{jk}]]$, where $a_1,\ldots,a_s\in[g]$. Lemma 18 implies that these elements all commute with t_{ik}, so $[t_{ik},(\alpha_j)_{|\Delta_{ik}}]=0$. In the same way, $[\alpha_i,\alpha_j]$ is regular at each diagonal Δ_{jk} $(k\neq i,j)$.

Let us now prove that $[\alpha_i,\alpha_j]$ is regular at Δ_{ij}. We will assume $i=1,j=2$. Let ω be a nonzero rational differential on C. $[\alpha_1,\alpha_2]=\frac{1}{\omega_1}[\alpha_1,\omega_1\alpha_2+\omega_2\alpha_1]$, so $[\alpha_1,\alpha_2]$ has at most simple poles at Δ_{12}, and $\mathrm{res}_{\Delta_{12}}[\alpha_1,\alpha_2]=[t_{12},\tilde\alpha_\omega]$. According to Proposition 13, $\tilde\alpha_\omega\in\mathbb{C}t_{12}+\mathrm{im}(\hat{\mathfrak{t}}_{g,n-1}\to\hat{\mathfrak{t}}_{g,n},x\mapsto x^{12,3,\cdots,n})$, therefore $[t_{12},\tilde\alpha_\omega]=0$, so $\mathrm{res}_{\Delta_{12}}[\alpha_1,\alpha_2]=0$.

All this implies that $[\alpha_i,\alpha_j]\in H^0(C^n,\mathrm{ad}\,\mathcal{P}_n\otimes K_C^{(i)}\otimes K_C^{(j)})$, and therefore identifies with an element $\beta\in H^0(\tilde C^n,K_{\tilde C}^{(i)}\otimes K_{\tilde C}^{(j)})\otimes\hat{\mathfrak{t}}_{g,n}[2]$ (where the degree in $\mathfrak{t}_{g,n}$ is given by $|x^k_a|=0$, $|y^k_a|=1$), such that $\gamma^k_a(\beta)=e^{\mathrm{ad}\,x^k_a}(\beta)$ for any $(k,a)\in[n]\times[g]$.

Recall that $\hat{\mathfrak{t}}_{g,n}$ is $\mathbb{N}$-graded by $|x^i_a|=1$. Decompose β according to this degree, so $\beta=\sum_{s\geq0}\beta_s$. Let us prove by induction that $\beta_s=0$. Assume that $\beta_{s'}=0$ for $s'<s$, then $\beta_s\in H^0(C^n,K_C^{(i)}\otimes K_C^{(j)})\otimes\mathfrak{t}_{g,n}[2][s]$. Since $H^0(C^n,K_C^{(i)}\otimes K_C^{(j)})\simeq H^0(C,K_C)^{\otimes2}$, there is a decomposition

$$\beta_s=\sum_{a,b\in[g]}\beta^{ab}_s\omega_a^{z_i}\omega_b^{z_j}.$$

For any $k\in[n]$, $(\gamma^k_a-1)\beta_{s+1}=[x^k_a,\beta_s]$.

If $k\neq i,j$, the r.h.s. is constant in the kth variable. If f is a regular function on $\tilde C$ such that $(\gamma_a-1)f=c_a$, where c_a are constants, then df is a univalued differential on $\tilde C$, i.e. an element of $H^0(C,K_C)$; as $\int_{\mathcal{A}_a}df=0$ for any $a\in[g]$, $df=0$, so f is constant. It follows that β_{s+1} is constant w.r.t. the kth variable.

If now ω is a regular differential on $\tilde C$ such that $(\gamma_a-1)\omega=\alpha_a$, where α_a are differentials, then $\sum_{a\in[g]}\int_{\mathcal{A}_a}\alpha_a=0$. Therefore $\sum_{a,b\in[g]}[x^i_a,\beta^{ab}_s]\omega_b^w=\sum_{a,b\in[g]}[x^j_b,\beta^{ab}_s]\omega_a^z=0$.

It follows that $(\beta^{ab}_s)_{a,b\in[g]}$ satisfies

$$\forall b\in[g],\sum_{a\in[g]}[x^i_a,\beta^{ab}_s]=0,\quad\forall a\in[g],\sum_{b\in[g]}[x^j_b,\beta^{ab}_s]=0,\quad[x^k_c,\beta^{ab}_s]=0$$

and belongs to $[V_i,V_j]$, where $V_i\subset\mathfrak{t}_{g,n}[1]$ is the linear span of $[x^i_{a_1},\ldots,[x^i_{a_s},y_b]]$, $[x^i_{a_1},\ldots,[x^i_{a_s},t_{ik}]]$, where $a_1,\ldots,a_s,b\in[g]$ and $k\neq i$.

Proposition 20 then implies that $\beta^{ab}_s=0$ for any a,b, therefore $\beta_s=0$. $\square$

Corollary 17. $\alpha_{KZ}\in\mathcal{F}_1^{hol}$.

This proves Theorem 3. In particular, α_{KZ} can be used for establishing the formality Theorem 1 and for constructing the extended morphism (9).

7. Postponed proofs: algebraic results on $\mathfrak{t}_{g,n}$

Lemma 18. *The following relations hold in* $\mathfrak{t}_{g,n}$ *:*

1) $t_{ji}=t_{ij}$, *if* $i\neq j$*;*
2) $[t_{ij},t_{ik}+t_{jk}]=0$, *if* i,j,k *are all different;*
3) $[t_{ij},t_{kl}]=0$, *if* i,j,k,l *are all different;*
4) $[v^i+v^j,t_{ij}]=0$, *if* $i\neq j$ *and* $v\in V$.

Proof. If $v, w \in V$, then $0 = [v^i, w^j] + [w^j, v^i] = \langle v, w\rangle t_{ij} + \langle w, v\rangle t_{ji} = \langle v, w\rangle(t_{ij} - t_{ji})$. This implies 1).

If $v \in V$ and $i \neq j$, then $0 = [v^j, \sum_a [x_a^i, y_a^i] + \sum_{k\neq i} t_{ik}] = \sum_a \langle v, x_a\rangle [t_{ij}, y_a^i] + \sum_a \langle v, y_a\rangle [x_a^i, t_{ij}] + [v^j, t_{ij}] = [v^i + v^j, t_{ij}]$, which implies 4).

If $w \in V$ and i, j, k are different, then $0 = [w^k, [v^i + v^j, t_{ij}]] = \langle v, w\rangle [t_{ki} + t_{kj}, t_{ij}]$, which implies 2).

If $v, w \in V$ and i, j, k, l are different, then $0 = [w^l, [v^k, t_{ij}]] = \langle w, v\rangle [t_{kl}, t_{ij}]$, which implies 3). $\square$

The Lie algebra $\mathfrak{t}_{g,n}$ therefore admits the presentation $\mathfrak{t}_{g,n} = \mathbb{L}(x_a^i, y_a^i, t_{ij}; i, j \in [n], a \in [g])/(R_0, R_1, R_2)$, where the relations are:

(R_0) $[x_a^i, x_b^j] = 0$ if $i \neq j$;

(R_1) $[x_a^i, y_b^j] = \delta_{ab} t_{ij}$ if $i \neq j$; $t_{ji} = t_{ij}$; $[x_a^i + x_a^j, t_{ij}] = [x_a^k, t_{ij}] = 0$ if i, j, k are distinct; $\sum_a [x_a^i, y_a^i] + \sum_{j:j\neq i} t_{ij} = 0$;

(R_2) $[y_a^i, y_b^j] = 0$ if $i \neq j$; $[y_a^i + y_a^j, t_{ij}] = [y_a^k, t_{ij}] = 0$ if i, j, k are distinct; $[t_{ij} + t_{ik}, t_{jk}] = [t_{ij}, t_{kl}] = 0$ if i, j, k, l are distinct.

Here $\mathbb{L}(V)$ is the free Lie algebra on a vector space V and if S is a set, then $\mathbb{L}(S) := \mathbb{L}(V)$, where $V = \mathbb{C}^{(S)}$ is the vector space with basis S.

If the generators are given the degrees $|x_a^i| = 0$, $|t_{ij}| = |y_a^i| = 1$, then the relations R_i are homogeneous of degree i ($i = 0, 1, 2$). According to [JW], the quotient $\mathbb{L}(x_a^i, y_a^i, t_{ij})/(R_0, R_1)$ is isomorphic to $\mathbb{L}(V) \rtimes \mathfrak{f}_g^{\oplus n}$, where V is the $\mathfrak{f}_g^{\oplus n}$-module with generators y_a^i, t_{ij} and relations: $x_a^i \cdot y_b^j = \delta_{ab} t_{ij}$ if $i \neq j$; $t_{ji} = t_{ij}$; $(x_a^i + x_a^j) \cdot t_{ij} = x_a^k \cdot t_{ij} = 0$ if i, j, k are distinct. This is an isomorphism of graded Lie algebras, where $\mathfrak{f}_g^{\oplus n}$ has degree 0 and V has degree 1. It follows that there is an isomorphism of $\mathfrak{f}_g^{\oplus n}$-modules

$$\mathfrak{t}_{g,n}[2] \simeq \mathbb{L}_2(V)/(R_2),$$

where $(R_2) \subset \mathbb{L}_2(V)$ is the $\mathfrak{f}_g^{\oplus n}$-submodule generated by R_2.

Define $\mathfrak{f}_g^{\oplus n}$-modules M_i, M_{ij} as follows. Set $F := U(\mathfrak{f}_g)$; this is the free associative algebra over generators $x_a, a \in [g]$. Denote also by F the left regular F-module (the action is $x \cdot f := xf$). There is a unique F-module morphism $F \to F^{\oplus g}$, $f \mapsto (fx_1, \dots, fx_g)$. We then define a F-module $M := \mathrm{Coker}(F \to F^{\oplus g})$. Define a $F^{\otimes 2}$-module $M_{12} := F^{\otimes 2}/$(left ideal generated by the $x_a \otimes 1 + 1 \otimes x_a, a \in [g]$), where $F^{\otimes 2}$ is viewed as the left regular $F^{\otimes 2}$-module. Then the $F^{\otimes 2}$-module M_{12} identifies with F, equipped with the action $(x \otimes y) \cdot f := xfS(y)$, where S is the antipode of F, under the map $F^{\otimes 2}/(\text{ideal}) \to F$, (class of $f \otimes g$) $\mapsto fS(g)$.

Set $M_i := p_i^*(M)$, where $p_i : F^{\otimes n} \to F$ is the morphism $p_i = \varepsilon^{\otimes i-1} \otimes \mathrm{id} \otimes \varepsilon^{\otimes n-i}$, and $M_{ij} := p_{ij}^*(M_{12})$, where $p_{ij} : F^{\otimes n} \to F^{\otimes 2}$ is given by $p_{ij} = \varepsilon^{\otimes i-1} \otimes \mathrm{id} \otimes \varepsilon^{\otimes j-i-1} \otimes \mathrm{id} \otimes \varepsilon^{\otimes n-j}$ if $i < j$, and $p_{ji} = p_{ij}$ ($\varepsilon : F \to \mathbb{C}$ is the counit of F). Then M_i and M_{ij} are $F^{\otimes n}$-modules, and $M_{ji} \simeq M_{ij}$.

Recall that $V_i \subset V$ is the linear span of the $[x_{a_1}^i, \dots, [x_{a_s}^i, y_b^i]], [x_{a_1}^i, \dots, [x_{a_s}^i, t_{ij}]], a_1, \dots, a_s, b \in [g]$, $j \neq i$, and may be viewed as the $\mathfrak{f}_g^{\oplus n}$-submodule of V generated by y_a^i, t_{ij}, $a \in [g], j \neq i$.

Proposition 19. *There are exact sequences of $\mathfrak{f}_g^{\oplus n}$-modules $0 \to \oplus_{i<j} M_{ij} \to V \to \oplus_i M_i \to 0$ and $0 \to \oplus_{j:j\neq i} M_{ij} \to V_i \to M_i \to 0$.*

Proof. The quotient of V by the submodule generated by the t_{ij} is clearly isomorphic to $\oplus_i M_i$. For any $i < j$, there is a unique morphism $M_{ij} \to V$, given by (class of $u \otimes v$) $\to u^{(i)} v^{(j)} \cdot t_{ij}$, which gives rise to a morphism $\oplus_{i<j} M_{ij} \to V$ such that $\oplus_{i<j} M_{ij} \to V \to \oplus_i M_i \to 0$ is exact.

It remains to prove that $\oplus_{i<j} M_{ij} \to V$ is injective. Set $\mathcal{M} := M_{12}^{\{(i,j)|i<j\}} \oplus F^{[n]\times[g]}$. Denote the map $M_{12} \to \mathcal{M}$ corresponding to (i, j) by $m \mapsto m_{ij}$ and the map $F \to \mathcal{M}$ corresponding to (i, a) by $m \mapsto m^{[i,a]}$. Let also $f \mapsto f^{(k)}$ be the morphism $F \to F^{\otimes n}$, $f \mapsto 1^{\otimes k-1} \otimes f \otimes 1^{\otimes n-k}$.

If $j > i$ and $m \in M_{12}$, we set $m_{ji} := (m^{21})_{ij}$, where $m \mapsto m^{21}$ is induced by the exchange of factors of $F^{\otimes 2}$.

There is a unique $F^{\otimes n}$-module structure over $\mathcal{M}$, such that $f^{(i)} \cdot m_{ij} = ((f \otimes 1)m)_{ij}$, $f^{(j)} \cdot m_{ij} = ((1 \otimes f)m)_{ij}$, $f^{(k)} \cdot m_{ij} = \varepsilon(f) m_{ij}$ if $k \neq i, j$, and $f^{(i)} \cdot m^{[i,a]} = (fm)^{[i,a]}$, $f^{(j)} \cdot m^{[i,a]} = (m \otimes \partial_a(f))_{ij}$ if $i \neq j$, where $\partial_a : F \to F$ is defined by $f = \varepsilon(f)1 + \sum_{a \in [g]} \partial_a(f) x_a$.

There is a unique morphism $p_i^*(F) \to \mathcal{M}$, given by $f \mapsto \sum_a (f x_a)^{[ia]} + \sum_{j: j \neq i} (f \otimes 1)_{ij}$. Set $\overline{\mathcal{M}} := \mathrm{Coker}(\oplus_i p_i^*(F) \to \mathcal{M})$. There is a unique morphism $V \to \overline{\mathcal{M}}$, such that $y_a^i \mapsto 1^{[ia]}$ and $t_{ij} \mapsto (1 \otimes 1)_{ij}$. The composed morphism $\oplus_{i<j} M_{ij} \to V \to \overline{\mathcal{M}}$ is injective as $(\oplus_{i<j} M_{ij}) \cap \mathrm{im}(\oplus_i p_i^*(F) \to \mathcal{M}) = \{0\}$. It follows that $\oplus_{i<j} M_{ij} \to V$ is injective, as claimed.

The image of the composed map $V_i \to V \to \oplus_j M_j$ is M_i, and the kernel of $V_i \to M_i$ is $V_i \cap (\oplus_{j<k} M_{jk}) = \oplus_{j: j \neq i} M_{ij}$. □

This exact sequence from Proposition 19 gives rise to a filtration $0 \subset V_0 \subset V_1 = V$, where $V_0 = \mathrm{gr}_0(V) = \oplus_{i<j} M_{ij}$ and $\mathrm{gr}_1(V) = \oplus_i M_i$. It induces a filtration on $X := \mathbb{L}_2(V)$, namely $0 \subset X_0 \subset X_1 \subset X_2 = X$, with $X_0 = \Lambda^2(V_0)$ and $X_1 = V_0 \wedge V_1$. Then $\mathrm{gr}(X) = \Lambda^2(\mathrm{gr}(V))$, explicitly

$$\mathrm{gr}_2(X) = \bigoplus_i \Lambda^2(M_i) \oplus \bigoplus_{i<j} M_i \otimes M_j,$$

$$\mathrm{gr}_1(X) = \bigoplus_{i;j<k} M_i \otimes M_{jk},$$

and

$$\mathrm{gr}_0(X) = \Lambda^2(X_0) = \bigoplus_{i<j} \Lambda^2(M_{ij}) \oplus \bigoplus_{i<j;k<l;(i,j)<(k,l)} M_{ij} \otimes M_{kl}$$

where the lexicographic order is implied.

The submodule $Y := (R_2) \subset X$ is then equipped with the induced filtration $0 \subset Y_0 \subset Y_1 \subset Y_2 = Y$, where $Y_0 := Y \cap X_0$, $Y_1 := Y \cap X_1$.

Recall that

$$Y = \sum_{i<j;a,b} F^{\otimes n} \cdot [y_a^i, y_b^j] + \sum_{i<j;a} F^{\otimes n} \cdot [y_a^i + y_a^j, t_{ij}] + \sum_{i<j;k \notin \{i,j\};a} F^{\otimes n} \cdot [y_a^k, t_{ij}]$$
$$+ \sum_{|\{i,j,k\}|=3} F^{\otimes n} \cdot [t_{ij}, t_{ik} + t_{jk}] + \sum_{|\{i,j,k,l\}|=4} F^{\otimes n} \cdot [t_{ij}, t_{kl}].$$

If $i < j$, then for $k \neq i, j$ and any c, $x_c^k \cdot [y_a^i, y_b^j] = \delta_{bc}[y_a^i, t_{kj}] - \delta_{ac}[y_b^j, t_{ik}]$ and $x_c^k \cdot [y_a^i + y_a^j, t_{ij}] = \delta_{ac}[t_{ik} + t_{jk}, t_{ij}]$. If $i < j$ and $k \notin \{i, j\}$, then for any $l \notin \{i, j, k\}$, $x_c^l \cdot [y_a^i, t_{jk}] = \delta_{ac}[t_{il}, t_{jk}]$. If $|\{i, j, k\}| = 3$ and $l \notin \{i, j, k\}$, then $x_a^l \cdot [t_{ij}, t_{ik} + t_{jk}] = 0$ and if $|\{i, j, k, l\}| = 4$ and $m \notin \{i, j, k, l\}$, then $x_a^m \cdot [t_{ij}, t_{kl}] = 0$. All this implies that

$$Y = \sum_{i<j;a,b} F_{\{i,j\}} \cdot [y_a^i, y_b^j] + \sum_{i<j;a} F_{\{i,j\}} \cdot [y_a^i + y_a^j, t_{ij}] + \sum_{i<j;k \notin \{i,j\};a} F_{\{i,j,k\}} \cdot [y_a^k, t_{ij}]$$
$$+ \sum_{|\{i,j,k\}|=3} F_{\{i,j,k\}} \cdot [t_{ij}, t_{ik} + t_{jk}] + \sum_{|\{i,j,k,l\}|=4} F_{\{i,j,k,l\}} \cdot [t_{ij}, t_{kl}] = \Sigma_1 + \cdots + \Sigma_5,$$

where for $S \subset [n]$, $F_S \subset F^{\otimes n}$ is $\otimes_{i=1}^n F_S(i)$, where $F_S(i) = F$ is $i \in S$ and $\mathbb{C}$ otherwise. Each of the summands is a $F^{\otimes n}$-module via the natural morphisms $F^{\otimes n} \to F_S$. Here $\Sigma_1, \ldots, \Sigma_5$ denote each of the summands.

We have obviously $\Sigma_4 + \Sigma_5 \subset Y_0$, $\Sigma_2 + \ldots + \Sigma_5 \subset Y_1$.

It follows from the second inclusion that if $K := \mathrm{Ker}(\oplus_{i<j} F_{\{i,j\}}^{[g]\times[g]} \to X/X_1)$ (the map being $(f_{i,j;ab})_{i,j;a,b} \mapsto \sum_{i<j;a,b} f_{i,j;a,b} \cdot [y_a^i, y_b^j]$), then $Y_1 = \mathrm{im}(K \to Y) + (\Sigma_2 + \cdots + \Sigma_5)$. While $X/X_1 = \mathrm{gr}_2(X) = \oplus_i \Lambda^2(M_i) \oplus \bigoplus_{i<j} M_i \otimes M_j$, the map defining K is the direct sum over the

pairs $(i,j), i<j$ of the maps $F^{[g]\times[g]}_{\{i,j\}} \to M_i \otimes M_j$ defined as $F^{[g]\times[g]}_{\{i,j\}} \simeq F^{\oplus g}\otimes F^{\oplus g} \to M^{\otimes 2} \simeq M_i \otimes M_j$. It follows that K is the direct sum over the pairs (i,j) of the kernels of each map corresponding to (i,j). This kernel is $\mathrm{im}(F^{\oplus g}\otimes F \oplus F \otimes F^{\oplus g} \to F^{\oplus g}\otimes F^{\oplus g})$, where the maps $F^{\oplus g}\to F^{\oplus g}$ are identity maps and $F \to F^{\oplus g}$ is $f \mapsto (fx_1,\dots,fx_g)$. Its image in Y_1 is therefore the $F_{\{i,j\}}$-submodule generated by all the $\sum_a x_a^i \cdot [y_a^i, y_b^j]$ $(b\in[g])$ and $\sum_b x_b^j \cdot [y_a^i, y_b^j]$ $(a\in[g])$. As these elements are equal to $[y_a^i + y_a^j, t_{ij}]$ and $[t_{ij}, y_b^i + y_b^j]$, these submodules are contained in Σ_2. It follows that

$$Y_1 = \Sigma_2 + \ldots + \Sigma_5.$$

Moreover,

$$\begin{aligned}\mathrm{gr}_2(Y) &= \mathrm{im}(Y \to X/X_1) = \mathrm{im}(\sum_{i<j;a,b} F_{\{i,j\}}\cdot [y_a^i, y_b^j] \to X/X_1)\\ &= \oplus_{i<j} M_i \otimes M_j. \end{aligned} \tag{13}$$

Since $\Sigma_4+\Sigma_5 \subset Y_0$ and $Y_1 = \Sigma_2+\cdots+\Sigma_5$,

$$Y_0 = \mathrm{Ker}(Y_1 \to X_1/X_0) = \Sigma_4+\Sigma_5+\mathrm{Ker}(\Sigma_2+\Sigma_3 \to X_1/X_0 = \mathrm{gr}_1(X)) = \Sigma_4+\Sigma_5+\mathrm{im}(K'\to Y),$$

where $K' = \mathrm{Ker}(\bigoplus_{i<j;k\neq i,j} F^g_{\{i,j,k\}} \oplus \bigoplus_{i<j} F^g_{\{i,j\}} \to X_1/X_0)$, the map being the sum of over i,j,k $(i<j;\ k\neq i,j)$ of

$$\varphi_{ijk} : F^g_{\{i,j,k\}} \simeq (F^{\otimes 3})^g \to \mathrm{gr}_1(X), \quad (f_a\otimes g_a \otimes h_a)_a \mapsto \sum_a f_a^{(i)} g_a^{(j)} h_a^{(k)} \cdot [y_a^k, t_{ij}]$$

and over i,j $(i<j)$ of

$$\psi_{ij} : F^g_{\{i,j\}} \simeq (F^{\otimes 2})^g \to \mathrm{gr}_1(X), \quad (f_a\otimes g_a)_a \mapsto \sum_a f_a^{(i)} g_a^{(j)} \cdot [y_a^i + y_a^j, t_{ij}].$$

The image of φ_{ijk} is contained in $M_k\otimes M_{ij}$, and the image of ψ_{ij} is contained in $(M_i\oplus M_j)\otimes M_{ij}$, therefore K' is the direct sum of the kernels of these maps.

The map φ_{ijk} is isomorphic to the tensor product $(F^g \to M)\otimes(F^{\otimes 2}\to M_{12})$, which is surjective and whose kernel is $\sum_a F^g \otimes F^{\otimes 2}(x_a\otimes 1 + 1\otimes x_a) + \mathrm{im}(F\to F^g)\otimes F^{\otimes 2}$. It follows that the image of $\mathrm{Ker}\,\varphi_{ijk}$ in Y is the $F^{\otimes n}$-submodule generated by $\sum_a x_a^i \cdot [y_a^i, t_{jk}] = -\sum_{l\neq i}[t_{il}, t_{jk}]$ and the $(x_b^j + x_b^k)\cdot [y_a^i, t_{jk}] = \delta_{ab}[t_{ij}+t_{ik}, t_{jk}]$ $(a,b\in[g])$, which is contained in $\Sigma_4+\Sigma_5$.

The map ψ_{ij} is isomorphic to the map

$$(F\otimes F)^g \to (M\otimes M_{12})^{\oplus 2} = ((F^g/F^{diag}\cdot(x_1,\dots,x_g))\otimes F)^{\oplus 2}, \tag{14}$$

$$(f_a\otimes g_a)_{a\in[g]} \mapsto (f_a^{(1)}\otimes f_a^{(2)} S(g_a))_{a\in[g]} \oplus (g_a^{(1)}\otimes g_a^{(2)} S(f_a))_{a\in[g]}.$$

The two maps $(F\otimes F)^g \to F^g\otimes F$ defined by these formulas are surjective, and the preimage of $F^{diag}\cdot(x_1,\dots,x_g)\otimes F$ under each of them is $(F^{diag}\otimes F)\cdot(x_1\otimes 1+1\otimes x_1,\dots,x_g\otimes 1 + 1\otimes x_g)$. It follows that $\mathrm{Ker}\,\psi_{ij}$ is the $F^{diag}_{\{i,j\}}$-submodule of $F^g_{\{i,j\}}$ generated by $\sum_a (x_a^i + x_a^j)$. Its image in Y is the $F^{\otimes n}$-submodule generated by $\sum_a (x_a^i + x_a^j)\cdot [y_a^i + y_a^j, t_{ij}] = -\sum_{k\neq i,j}[t_{ik}+t_{jk}, t_{ij}]$ and is therefore contained in $\Sigma_4+\Sigma_5$. Therefore

$$Y_0 = \Sigma_4 + \Sigma_5 + \mathrm{im}(K'\to Y) = \Sigma_4+\Sigma_5.$$

It follows also that the two maps from $(F^g\otimes F)/(F^{diag}\otimes F)\cdot(x_1\otimes 1 + 1\otimes x_1,\dots,x_g\otimes 1+1\otimes x_g)$ to $M_i\otimes M_{ij}$ and $M_j\otimes M_{ij}$ derived from (14) are isomorphisms (in particular, $M_i\otimes M_{ij}$ and $M_j\otimes M_{ij}$ are isomorphic). The image of ψ_{ij} is then a diagonal submodule $(M\otimes M_{12})_{ij} \subset (M_i\oplus M_j)\otimes M_{ij}$. Then

$$\mathrm{gr}_1(Y) = \bigoplus_{i<j;k\neq i,j} M_k\otimes M_{ij} \oplus \bigoplus_{i<j} (M\otimes M_{12})_{ij}. \tag{15}$$

Recall that

$$\mathrm{gr}_0(X) = \bigoplus_{|\{i,j,k,l\}|=4; i<j; k<l; i<k} M_{ij} \otimes M_{kl} \oplus \bigoplus_{i<j<k} (M_{ij} \otimes M_{ik} \oplus M_{ij} \otimes M_{jk} \oplus M_{ik} \otimes M_{jk}).$$

$\Sigma_4 + \Sigma_5 \subset \mathrm{gr}_2(X)$ is compatible with this decomposition, so

$$\mathrm{gr}_0(Y) = \Sigma_4 + \Sigma_5 = \bigoplus_{|\{i,j,k,l\}|=4; i<j; k<l; i<k} M_{ij} \otimes M_{kl}$$
$$\oplus \bigoplus_{i<j<k} \mathrm{im}\big(F_{\{i,j,k\}} \cdot [t_{ij}, t_{ik} + t_{jk}] + F_{\{i,j,k\}} \cdot [t_{ik}, t_{ij} + t_{jk}] + F_{\{i,j,k\}} \cdot [t_{jk}, t_{ij} + t_{ik}]$$
$$\to M_{ij} \otimes M_{ik} \oplus M_{ij} \otimes M_{jk} \oplus M_{ik} \otimes M_{jk}\big). \tag{16}$$

The filtration of X induces a filtration on $\mathfrak{t}_{g,n}[2] = X/Y$, whose associated graded is according to (13), (15) and (16)

$$\mathrm{gr}_2\, \mathfrak{t}_{g,n}[2] = \oplus_i \Lambda^2(M_i), \tag{17}$$

$$\mathrm{gr}_1\, \mathfrak{t}_{g,n}[2] = \oplus_i M_i \otimes M_{ij}, \tag{18}$$

$$\mathrm{gr}_0\, \mathfrak{t}_{g,n}[2] = \oplus_{i<j<k} M_{ijk}, \tag{19}$$

where M_{123} is the $F^{\otimes 3}$-module with generator ω_{123} and relations $(x_a^1 + x_a^2 + x_a^3) \cdot \omega_{123} = 0$ for $a \in [g]$, $\omega_{\sigma(1)\sigma(2)\sigma(3)} = \varepsilon(\sigma)\omega_{123}$ for $\sigma \in S_3$, and M_{ijk} is its pull-back under the morphism $F^{\otimes n} \to F^{\otimes 3}$ associated to (i,j,k).

Proposition 20. *Let $(\beta_{ab})_{a,b\in[g]}$ be a family of elements of $[V_i, V_j]$ such that: (a) each β_{ab} commutes with the x_c^k, $c \in [g]$, $k \neq i,j$; (b) $\forall b \in [g]$, $\sum_{a\in[g]}[x_a^i, \beta_{ab}] = 0$; (c) $\forall a \in [g]$, $\sum_{b\in[g]}[x_b^j, \beta_{ab}] = 0$. Then $\beta_{ab} = 0$ for any a, b.*

Proof. Recall that the $F^{\otimes n}$-module $Z := \mathfrak{t}_{g,n}[2]$ admits a filtration $\{0\} \subset Z_0 \subset Z_1 \subset Z_2 = Z$.

Lemma 21. $[V_i, V_j] \subset Z_1$.

Proof of Lemma. This means that the map $[V_i, V_j] \to \mathrm{gr}_2\, \mathfrak{t}_{g,n}[2]$ is zero. The image of this map is the same as that of $V_i \otimes V_j \to \mathbb{L}_2(V) \to \mathrm{gr}_2\, \mathbb{L}_2(V) \to \mathrm{gr}_2\, \mathfrak{t}_{g,n}[2]$. The image of $V_i \otimes V_j \to \mathbb{L}_2(V) \to \mathrm{gr}_2\, \mathbb{L}_2(V) \simeq \bigoplus_\alpha \Lambda^2(M_\alpha) \oplus \bigoplus_{\alpha<\beta} M_\alpha \otimes M_\beta$ is $M_i \otimes M_j$, whereas $\mathbb{L}_2(V) \to \mathrm{gr}_2\, \mathfrak{t}_{g,n}[2]$ is the natural projection on $\bigoplus_\alpha \Lambda^2(M_\alpha)$. It follows that the image of $V_i \otimes V_j \to \mathrm{gr}_2\, \mathfrak{t}_{g,n}[2]$ is zero, as wanted. □

Let $\mathcal{C}$ be the category of $F^{\otimes n}$-modules M equipped with a $\mathbb{N}$-grading compatible with the $\mathbb{N}$-grading of $F^{\otimes n}$ given by $|x_a^i| = 1$, and where the morphisms are restricted to be of degree zero. This is a tensor subcategory of the category of all $F^{\otimes n}$-modules. The modules M_α ($\alpha \in [g]$), $M_{\alpha\beta}$ ($\alpha < \beta \in [g]$), $M_{\alpha\beta\gamma}$ ($\alpha < \beta < \gamma \in [g]$) are objects in $\mathcal{C}$.

Let us say that the $F^{\otimes n}$-module M has property (P) if the map

$$M^{[g]\times[g]} \to M^{[g]^3 \times([n]-\{i,j\})} \oplus M^{[g]} \oplus M^{[g]},$$
$$(\beta_{ab})_{a,b\in[g]} \mapsto (x_c^k \cdot \beta_{ab})_{a,b,c\in[g];k\neq i,j} \oplus (\sum_{c\in[g]} x_c^i \cdot \beta_{ca})_{a\in[g]} \oplus (\sum_{c\in[g]} x_c^j \cdot \beta_{ac})_{a\in[g]}$$

is injective.

Lemma 22. *1) If $M \subset N$ is an inclusion of $F^{\otimes n}$-modules and N has (P), then M has (P).*

2) If $M = M^0 \supset M^1 \supset \cdots \supset M^s = \{0\}$ is a sequence of inclusions of $F^{\otimes n}$-modules and if each M^i/M^{i+1} has (P), then M has (P).

3) If M, N are objects of $\mathcal{C}$ and M or N has (P), then $M \otimes N$ has (P).

4) The modules $M_{\alpha\beta}$ ($\alpha < \beta$) and $M_{\alpha\beta\gamma}$ ($\alpha < \beta < \gamma$) have (P).

Proof of Lemma. 1) and 2) are immediate. Set $S := [g]\times[g]$, $T := [g]^3\times([n]-\{i,j\})\sqcup[g]\sqcup[g]$, then the map involved in property (P) has the form $M^S \to M^T$. If M is an object of $\mathcal{C}$, this map decomposes as a direct sum of maps $M_i^S \to M_{i+1}^T$ for $i\geq 0$, where $M = \oplus_{i\geq 0}M_i$ is the decomposition of M. Let M, N be objects of $\mathcal{C}$ with decompositions $M=\oplus_{i\geq 0}M_i$, $N=\oplus_{i\geq 0}N_i$ and with property (P). The map involved in property (P) for $M\otimes N$ is the direct sum over $k\geq 0$ of maps $f:(\oplus_{i+j=k}M_i\otimes N_j)^S \to (\oplus_{i+j=k+1}M_i\otimes N_j)^T$, where each component (i,j) of the source is mapped to components $(i+1,j)$ and $(i,j+1)$ of the target. It follows that f is compatible with the decreasing filtration of both sides, for which $F^\alpha((\oplus_{i+j=l}M_i\otimes N_j)^X) = (\oplus_{i+j=l;j\geq\alpha}M_i\otimes N_j)^X$ $(l=k,k+1;\ X=S,T)$, and the associated graded map is $g\otimes\mathrm{id}: M^S_{k-\alpha}\otimes N_\alpha \to M^T_{k+1-\alpha}\otimes N_\alpha$, where g is the restriction of the map attached to M to degree $k-\alpha$. As this map is injective, so is f. This proves 3).

The $F^{\otimes n}$-module $M_{\alpha\beta}$ identifies with F, equipped with the action $x^{(k)}\cdot f := \varepsilon(x)f$ $(k\neq\alpha,\beta)$, $x^{(\alpha)}\cdot f := xf$, $x^{(\beta)}\cdot f := fS(x)$ for $x\in F$. The actions of x_c^α and of x_c^β on $M_{\alpha\beta}$ are therefore injective. If $(\alpha,\beta)\neq(i,j)$, this implies that $M_{\alpha\beta}$ has property (P). If now $(f_{ab})_{a,b\in[g]\times[g]}\in M_{ij}^{[g]\times[g]}\simeq F^{[g]\times[g]}$ is such that for any $b\in[g]$, $\sum_c x_c^i\cdot f_{cb}=0$, then $\sum_c x_c f_{cb}=0$, which implies, as F is a free algebra, that $f_{ab}=0$ for any a,b. So M_{ij} has property (P).

$M_{\alpha\beta\gamma}$ is a subobject of the object $\overline{M}_{\alpha\beta\gamma}$ of $\mathcal{C}$ defined as $F^{\otimes 3}/\sum_{a\in[g]}F^{\otimes 3}(x_a^{(1)}+x_a^{(2)}+x_a^{(3)})$, where the action of $F^{\otimes n}$ is given by $x^{(k)}\cdot f=\varepsilon(x)f$ $(k\notin\{\alpha,\beta,\gamma\})$, $x^{(\alpha)}\cdot f=(x\otimes 1\otimes 1)f$, $x^{(\beta)}\cdot f=(1\otimes x\otimes 1)f$, $x^{(\gamma)}\cdot f=(1\otimes 1\otimes x)f$. This module identifies via $f\otimes g\otimes h\mapsto fS(h^{(1)})\otimes gS(h^{(2)})$ with $F^{\otimes 2}$, equipped with the following action of $F^{\otimes n}$: $x^{(k)}\cdot f=\varepsilon(x)f$ $(k\notin\{\alpha,\beta,\gamma\})$, $x^{(\alpha)}\cdot f=(x\otimes 1)f$, $x^{(\beta)}\cdot f=(1\otimes x)f$, $x^{(\gamma)}\cdot f=f(S\otimes S)(x)$. Choose k in $\{\alpha,\beta,\gamma\}$ different from i or j. Since $F^{\otimes 2}$ is a domain, the above description shows that the action of x_c^k on $\overline{M}_{\alpha\beta\gamma}$ is injective for any c. This implies that $\overline{M}_{\alpha\beta\gamma}$ has (P), and therefore that $M_{\alpha\beta\gamma}$ also has (P). □

End of proof of Proposition 20. Z_1 admits a filtration $Z_0\subset Z_1$, where both $Z_1/Z_0=\mathrm{gr}_1\,\mathfrak{t}_{g,n}[2]$ and $Z_0=\mathrm{gr}_0\,\mathfrak{t}_{g,n}[2]$ have property (P) by virtue of (18), (19) and Lemma 22, 3) and 4). By the same Lemma, 2), Z_1 has therefore property (P). $[V_i,V_j]\subset Z_1$ by Lemma 21, so Lemma 22, 1) implies that $[V_i,V_j]$ has property (P), as claimed. □

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